Introduction to Audio and Music Engineering

Lecture 2

• A few mathematical prerequisites
• Limits and derivatives
• Simple harmonic oscillators
• Strings, Oscillations & Modes
Limits and Derivatives

Slope = \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0} = \frac{\Delta f}{\Delta x}

Make \Delta x \to 0

\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \equiv \frac{d}{dx} f(x) \bigg|_{x_0} \equiv f'(x_0)
The tangent line at $x_0$ is given by:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Where $f'(x_0)$ is the derivative of $f$ at $x_0$. The point $(x_0, f(x_0))$ is on the graph, and the tangent line is the best linear approximation of the function near $x_0$. The graph shows the function $f(x)$ with its tangent line at $x_0$. The derivative $f'(x_0)$ is the slope of the tangent line.
A few simple derivatives we will need ...

\[
d \frac{x^n}{dx} = nx^{n-1} \quad \frac{d}{dx} \sin(x) = \cos(x)
\]

\[
d \frac{\text{Const}}{dx} = 0 \quad \frac{d}{dx} \cos(x) = -\sin(x)
\]

\[
d \frac{x}{dx} = 1 \quad \frac{d}{dx} e^x = e^x
\]

Product rule: \[
\frac{d}{dx} \left[ f(x) \cdot g(x) \right] = f(x) \cdot g'(x) + f'(x) \cdot g(x)
\]

\[
\frac{d}{dx} \left[ x \sin(x) \right] = x \cdot \cos(x) + 1 \cdot \sin(x)
\]

Chain rule: \[
\frac{d}{dx} \left[ f(g(x)) \right] = f'(g(x)) \cdot g'(x)
\]

\[
\frac{d}{dx} \left[ \sin(x^2) \right] = \cos(x^2) \cdot 2x \quad \frac{d}{dx} e^{ax} = ae^{ax}
\]
Question

What is the derivative w.r.t. \( x \) of: \( 2x \sin(2x) \)

a) \( 2x \cos(2x) \)

b) \( 2 \sin(2x) + 2x \cos(2x) \)

c) \( 2 \sin(2x) + 4x \cos(2x) \)

d) \( 2 \cos(2x) + 4x \sin(2x) \)
What’s so special about e?

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 ...

\[ \frac{d}{dx} e^x = e^x \]

The only exponential function \( a^x \) where the slope of the function equals the value of the function at every point.

Bacteria colony growth: Let’s say that there is a colony of bacteria growing in a petri dish and that the rate of increase of the number of bacteria is 2 times the number already present. How does the population grow over time?

Let \( y(t) \) = number of bacteria at time \( t \), and \( y(t=0) = N_0 \), (initial condition)

then \[ \frac{d}{dt} y = 2 \cdot y \] solution is … \[ y(t) = A e^{2t} \]

and since \( y(t=0) = N_0 \) \( \rightarrow \) \( A = N_0 \)

\[ y(t) = N_0 e^{2t} \]
Derivatives of sin, cos

\[
\frac{d}{dt} \sin(t) = \cos(t)
\]

\[
\frac{d}{dt} \cos(t) = -\sin(t)
\]

\[
\frac{d}{dt} (-\sin(t)) = -\cos(t)
\]

\[
\frac{d}{dt} (-\cos(t)) = \sin(t)
\]
Simple Harmonic Oscillator

\[ x(t) = x_{\text{max}} \sin(t) \]

\[ v(t) = v_{\text{max}} \cos(t) \]
### Simple Harmonic Oscillator

**Newton’s 2nd Law**

\[
\frac{d}{dt} x = v, \text{ velocity}
\]

\[
\frac{d}{dt} v = a, \text{ acceleration}
\]

**Hooke’s Law**

\[
F = -kx
\]

\[
m \frac{d^2x}{dt^2} = -kx
\]

\[
\frac{d^2x}{dt^2} = -\frac{k}{m} x
\]

Let \( \omega^2 \equiv \frac{k}{m} \), so

\[
\frac{d^2x}{dt^2} = -\omega^2 x
\]

Can we find a function that satisfies this differential equation?

\[
x = x_0 \sin(\omega t)
\]

\[
\frac{d}{dt} x = x_0 \omega \cos(\omega t)
\]

\[
\frac{d^2}{dt^2} x = -x_0 \omega^2 \sin(\omega t)
\]

\[
\text{so } x(t) = x_0 \sin(\omega t)
\]

where \( \omega \equiv \sqrt{\frac{k}{m}} \)

It works!
Frequency and Period

\[ x(t) = x_0 \sin(\omega t) \]

where \( \omega \equiv \sqrt{\frac{k}{m}} \)

Sine repeats every \( 2\pi \)

\[ \sin(\omega t) \]

\[ \omega t = 0 \]

\[ \omega t = 2\pi \]

\[ T = \frac{2\pi}{\omega} \]

Period (seconds per cycle)

\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]

Frequency (cycles per second)

Angular frequency (radians per second)

\[ \omega = 2\pi f \]
Question

What is approximate frequency (in Hertz) of a simple harmonic oscillator of mass 1 kg with a spring constant of 9 Nts/m?

a) 2 Hz  

b) 0.5 Hz  

c) 9 Hz  

d) 1/9 Hz
Other systems that display simple harmonic oscillation

Simple pendulum

\[ \omega \equiv \sqrt{\frac{g}{l}} \quad f \equiv \frac{1}{2\pi} \sqrt{\frac{g}{l}} \]

Helmholtz resonator

mass \( m = \rho S L \)

Spring

\[ k = \rho \frac{S^2 c^2}{V} \quad c = \text{sound velocity} \]

\[
\begin{align*}
&c = 340 \text{ m/sec} \\
&S = \pi \times 10^{-4} \text{ m}^2 \\
&V = 100 \text{ cc} = 10^{-4} \text{ m}^3 \\
&L = 3 \times 10^{-2} \text{ m} \\
&f = 500 \text{ Hz}
\end{align*}
\]
Electrical Oscillator:

\[ \omega = \frac{1}{\sqrt{LC}} \]

\[ f = \frac{1}{2\pi} \frac{1}{\sqrt{LC}} \]

Resonant frequency

Capacitor

Inductor

“spring”

“mass”