Shear wave speed recovery using moving interference patterns obtained in sonoelastography experiments

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Abstract. Two new experiments have been created to characterize the elasticity of soft tissue using sonoelastography. In both experiments the spectral variance image displayed on a GE Logic 700 ultrasound machine shows a moving interference pattern that moves at a very small fraction of the shear wave speed. The goal of this paper is to devise and test algorithms to calculate the speed of the moving interference pattern using the arrival times of these same patterns. A geometric optics expansion is used to obtain Eikonal equations relating the arrival times of the moving interference pattern to the moving interference pattern speed and then to the shear wave speed. A cross-correlation procedure is employed to find the arrival times; and an inverse Eikonal solver called the level curve method computes the speed of the interference pattern. The algorithm is tested on data from a phantom experiment performed at the University of Rochester Center for Ultrasound.

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I. Introduction

The target in this paper is to produce an image of tissue where the imaging functional
is a measure of shear stiffness. This problem has been addressed for over ten years by a
number of researchers, for example (Bercoff *et al.* 2001, Catheline *et al.* 1999, Greenleaf
motivated by the fact that shear stiffness is the tissue elastic property that is felt in a
palpation exam. Three types of experiments have emerged.

- Static experiment: the tissue is compressed; (Konofagou *et al.* 1998, 2000a, 2000b,
  2000c);

- Transient experiments: (1) a wave is initiated with a line source on the boundary,
or in the interior, (Bercoff *et al.* 2002, 2004), and a wave with a front propagates
away from the source; (2) a wave is initiated at a point, (Nightingale *et al.* 2002,
2003), and propagates away from the source; and (3) a travelling wave is produced
by harmonic excitation at two different points each excited at two different, but
nearby, frequencies, (Wu *et al.* 2004, 2006)

- Dynamic excitation: (1) a time harmonic excitation made on the boundary creates
a time harmonic wave in the tissue (Gao *et al.* 1995, Taylor *et al.* 2000, Wu *et al.*
2002); and (2) a time harmonic excitation in the interior, (Greenleaf and Fatemi
1998), creates a time harmonic radiating wave;

For most of these experiments interior displacement on a fine grid of points in an
imaging plane is measured with ultrasound or MR. In most cases where an excitation
occurs, the excitation is low frequency (50-200Hz); in (Greenleaf and Fatemi 1998) an
interior point source excitation at a few KHz is used.
In (McLaughlin and Renzi 2006a, 2006b) the authors developed the Arrival Time algorithm for the transient elastography experiment developed in the laboratory of M Fink, (Catheline et al. 1999, Sandrin et al. 2002a, 2002b). For this experiment a line source, with central frequency (50-200Hz), on the boundary initiates a shear wave propagating into the interior. The Arrival Time algorithm recovers the shear stiffness in the imaging plane from the time/space position of the wave front. Note that an important feature in the ultrafast imaging system developed by Fink, et al, is that the frame rate can be up to 10,000 frames/sec so that the propagating wave is sampled sufficiently to locate wave displacement features propagating into the medium.

Algorithms and images created by data obtained from a new sonoelastography experiment, developed by Wu and Parker at the University of Rochester, are the focus of this paper. In the previous sonoelasticity experiments, a time harmonic excitation on the boundary creates a time harmonic wave. The amplitude of the internal tissue vibrations, but not the phase, is displayed on a GE Logic 700 Doppler ultrasound machine after the tissue has reached steady state, see (Huang et al. 1990). In (Gao et al. 1995, Taylor et al. 2000, Wu et al. 2002) inverse methods are not applied to obtain shear stiffness images, but rather images are made of the vibrational amplitude itself. It is shown that regions of small vibrations indicate stiff inclusions. Examples are shown in (Gao et al. 1995, Taylor et al. 2000, Wu et al. 2002) when small vibrational amplitude created by sonoelastography can identify abnormal tissue when conventional ultrasound can’t.

More recently, two novel new experiments have been proposed where the image plane contains a very slow moving travelling wave. The first experiment, (Wu et al. 2004, 2006), using the terminology of the authors, is called the crawling wave experiment. There two time harmonic excitations are created on the opposite sides of the tissue. When the two excitations oscillate at the same frequency, the vibrational amplitude of
Shear stiffness recovery using interference patterns

the sum of these two waves forms a static interference pattern. In (Wu et al. 2004), the shear wave speed in a homogeneous medium is estimated using the distance between the stripes in the interference pattern; the authors refer to this as LFE, low frequency estimation. Now, when the two excitations oscillate at nearby frequencies, the stripes in the interference pattern move across the medium. Most importantly, when the two excitation frequencies are very close, this moving interference pattern travels at a small fraction of the shear wave speed. A propagating shear wave with velocity approximately $3m/sec$ travels much too quickly for the amplitude and phase of its displacement to be adequately imaged with the frame rate on the GE Logic 700 Doppler ultrasound machine, used for the University of Rochester experiment. However, the interference patterns created by two close but not equal vibration frequencies travel much slower so the moving vibrational amplitude of the interference pattern can be imaged, by displaying the Doppler spectral variance (see Section II). Furthermore, because the wave can be slowed so dramatically, the effective sampling rate (the sampling rate divided by the wave speed) has the possibility to be equivalent to that in the superfast imaging system designed in the laboratory of M. Fink. The GE Logiq 700 ultrasound machine yields (a color coded image whose color bars are closely related to) the interior vibrational amplitude data for the radial component of the elastic vector displacement as a function of space and time. The way we use this data is to treat one of the interference pattern stripes as a wave packet and determine the arrival times of one of the wave packets at each point in the medium. The ultimate goal then will be to find the speed of the moving interference pattern using the arrival times of the wave packet. When the medium is homogeneous, see (Wu et al. 2004), the speed of the moving interference pattern is a good indicator of the shear wave speed. However, note that in general, for this experiment, the relationship between the speed of the moving interference pattern and the shear wave speed is quite complicated. We explain this
In the second experiment, which using the University of Rochester terminology, (Wu et al. 2006), we will refer to as the holographic wave experiment, only one time harmonic excitation is made in the tissue. A second time harmonic wave is created by oscillating the ultrasound transducers that record the measurements. While this creates no vibrations in the tissue, the color coded display on the GE Logic 700 depends on the motion of the tissue relative to the motion of the ultrasound transducers. Once again, if the oscillations of the transducer and the excitation are at the same frequency, a standing interference pattern is observed. If instead two nearby frequencies are used, the GE Logic 700 displays the Doppler spectral variance which is seen as an interference pattern moving slowly across the medium at a small fraction of the shear wave speed velocity. Here the moving spectral variance image is related to the amplitude of the relative motion, (see Section II). For this second experiment we again determine the arrival times of one of the moving interference stripes; then we use the arrival times to calculate the speed of the moving interference patterns. In this case a simple calculation yields the shear wave speed at each point of the tissue in the imaging plane. There are two advantages to the experiment with only one tissue excitation source. The first is that excitations are not needed at opposite ends of the medium, as this may not be easily achievable in vivo. Secondly, there is a simple linear relationship between the speed of the moving interference pattern and the quantity of interest (shear wave speed). While the wave speed could be reconstructed using low frequency estimation, (Wu et al. 2004, 2006); here we obtain an improved image using the inverse level curve method, which is one of the versions of the Arrival Time Algorithm, see (McLaughlin and Renzi 2006b).

To accomplish our goal for both of these experiments, we will first show that: (1) the phase of the measured vibrational amplitude can be interpreted as the first arrival times of one of the stripes in the moving interference pattern; and (2) these arrival times
satisfy the Eikonal equation which relates the arrival times to the speed of the moving interference pattern. We use this information to develop an algorithm composed of two sub-algorithms.

- Finding the arrival times of a stripe or wave packet, in the moving interference pattern from the vibrational amplitude data;
- Using the Eikonal equation and the level curve method to find the speed of the moving interference patterns.

The first sub-algorithm finds the arrival time at each point, \( \vec{x} \), by finding the time delay, \( \delta t \), that maximizes the correlation between the time trace of the vibrational amplitude, \( u(\vec{x}, t - \delta t) \), and a reference signal. Then \( \delta t = T(\vec{x}) \), the arrival time at \( \vec{x} \). What is different here, compared to the cross-correlation procedure employed for transient elastography, is that since the signals are cyclical the cross-correlation function has many peaks of almost equal value. We will use the arrival times at nearby points to generate a guess, and find the local maximum, which is well defined, in the cross-correlation function near that guess.

The second sub-algorithm, takes as input the arrival times output from the first sub-algorithm, and uses the Eikonal equation, \( F(\vec{x}) = |\nabla T|^{-1} \), to find the speed, \( F(\vec{x}) \) of the moving interference patterns. Computing \( |\nabla T|^{-1} \) directly can lead to large outliers in the recovered speed because derivatives of the arrival times appear in the denominator, and noise in the data can push the denominator close to zero. The level curve algorithm removes this problem by developing a linear relation, without linearizing, between approximations to the wave speed and distances between level sets of the arrival time surface, \( \{(\vec{x}, t)|t = T(\vec{x})\} \). This linear relation leads to difference schemes where the derivatives of \( T \) are in the numerator.

The rest of this paper is composed as follows: First, the experimental setups for each experiment are illustrated in Section II. Next, the mathematical model for each
II. Experimental setups and the Data

The experimental apparatus for the crawling and holographic wave experiments are shown in Figure 1 A and B. In the crawling wave experiment, two vibration sources are placed on opposite ends of the testing sample and oscillate in the same direction parallel to the side surface. These sources oscillate at nearby, but not equal frequencies. An ultrasound probe is placed on top of the sample and the GE Logic 700 Doppler ultrasound machine, which has been reprogrammed to display Doppler spectral variance, presents a 16 color display of a wave slowly moving across the screen. This wave amplitude display is related to the radial components, \( A_r \sin(\omega t - \omega_1 \varphi_1) \),
$B_s \sin(\omega_2 t - \omega_1 \phi_2)$, of the two waves each emanating from one of the sources. To see
this, see (Huang, et al. 1990), the back scattered ultrasound signal can be represented
as
\[
s(t) = \cos \left( \omega_o t + \frac{4\pi}{\lambda_o} (A_r \sin(\omega_1 t - \phi_1) + B_r \sin(\omega_2 t + \phi_2)) \right)
\]
where $\omega_o$ is the ultrasound frequency and $\lambda_o$ the wavelength of the ultrasound wave.
Using basic trigonometry we can rewrite $s(t)$ as
\[
s(t) = \cos \left( w_o t + \frac{4\pi}{\lambda_o} G \sin \left( t(\omega_1 + \omega_2)/2 - (\omega_1 \phi_1 - \omega_2 \phi_2)/2 + \alpha \right) \right)
\]
where
\[
G = \left[ (A_r + B_r)^2 \cos^2(t(\omega_1 - \omega_2)/2 - (\omega_1 \phi_1 + \omega_2 \phi_2)/2) \right.
\]
\[\left. + (A_r - B_r)^2 \sin^2(t(\omega_1 - \omega_2)/2 - (\omega_1 \phi_1 + \omega_2 \phi_2)/2) \right]^{1/2}
\]
and
\[
\alpha = \tan^{-1} \left[ \left( \frac{A_r - B_r}{A_r + B_r} \right) \tan \left( t(\omega_1 - \omega_2)/2 - (\omega_1 \phi_1 + \omega_2 \phi_2)/2 \right) \right].
\]
Noting that since $\omega_1 - \omega_2 \sim 1Hz$ and $\omega_1 + \omega_2 \sim 400Hz$, $G$ and $\alpha$ are slowly varying
and can be considered as stationary during the time interval for calculating the spectral
variance. From (Huang, et al. 1990) the Doppler spectral variance, which is displayed on
the GE Logic 700 screen is proportional to $G^2$ which is, again using basic trigonometry,
\[
G^2 = A_r^2 + B_r^2 + 2A_rB_r \cos \left( t(\omega_1 - \omega_2) - (\omega_1 \phi_1 + \omega_2 \phi_2) \right).
\]

The holographic wave experimental setup is similar but the interpretation of the
data is different. Here only one vibrating source is placed on the phantom or tissue
and the ultrasound transducer itself vibrates at a nearby frequency. A gel is applied to
the transducer so that no wave motion created by the transducer propagates into the
sample. Letting $B \sin(\omega_2 t)$ be the vibration of the ultrasound transducers and noting
that the vibration perceived by the back scattered signal is the vibration of the tissue
relative to the vibration in the transducer we see that the backscattered signal can be
Shear stiffness recovery using interference patterns

represented as

\[ s(t) = \cos \left( \omega_0 t + \frac{4\pi}{\lambda_0} (A_\alpha \sin(\omega_1 t - \phi_1) - B \sin(\omega_2 t)) \right) \]

\[ = \cos \left( \omega_0 t + \frac{4\pi}{\lambda_0} \tilde{G} \sin(t(\omega_1 + \omega_2)/2 - \omega_1 \phi_1/2 + \tilde{\alpha}) \right) \]

where \( \tilde{\alpha} = \tan^{-1} \left[ \frac{A_\alpha + B}{A_\alpha - B} \tan^{-1}(t(\omega_1 - \omega_2)/2 - \omega_1 \phi_1/2) \right] \)

and

\[ \tilde{G}^2 = A_\alpha^2 + B^2 - 2A_\alpha B \cos(t(\omega_1 - \omega_2) - \omega_1 \phi_1) \]

and \( \tilde{G}^2 \) is proportional to the displayed Doppler spectral variance.

Note that again using basic trigonometry, for both experiments we have the important identities for the amplitude of the sum (or difference) of the complexification of the induced vibrations

\[
G^2 = \left| A_\alpha e^{i\omega_1 (t - \phi_1)} + B e^{i\omega_2 (t - \phi_2)} \right|^2
\]

\[
\tilde{G}^2 = \left| A_\alpha e^{i\omega_1 (t - \phi_1)} - B e^{i\omega_2 t} \right|^2
\]

These identities will be important in Section IV.

III. Mathematical model

Here we discuss the mathematical model used for the one and two frequency experiments.

Let \( \Omega \) be the domain containing the tissue to be imaged, and the measurement period \( \hat{T} > 0 \) be fixed. Because the displacements generated from the vibrators are small (on the order of microns), we will use a linear system of differential equations as a mathematical model. Since stiffness is an elastic property, the proper model for this experiment is the linear equations of elasticity. In this initial investigation we will further assume that the medium is isotropic. For an isotropic model the material parameters that describe the medium are the Lamé parameters, \( \lambda \) and \( \mu \), and the density, \( \rho \). The vector elastic
displacement, $\vec{u}$, is then governed by the following system of equations:

\[(\lambda u_{j,j},i + (\mu(u_{i,j} + u_{j,i})),j - \rho u_{i,tt} = 0 \text{ in } \Omega \times (0, T). \quad (1)\]

**IV. Equations for the imaging functionals**

In the crawling wave experiments, the displacement $\vec{u}$ can be broken up into two parts, $\vec{u} = \vec{u}^1 + \vec{u}^2$, where $\vec{u}^1$ is the displacement from the first vibration source and $\vec{u}^2$ is the displacement from the second vibration source. Naturally, $\vec{u}, \vec{u}^1, \text{ and } \vec{u}^2$ all satisfy equation (1). The goal here is to derive a relationship between the phase of the measured vibration amplitude and the shear wave speed.

To accomplish this goal we first assume that $\vec{u}, \vec{u}^1, \vec{u}^2$ represent the complexification of the corresponding displacements. We will use the geometric optics approximation, (Ji and McLaughlin 2004), for the complexification of the displacement, $\vec{u}^1$, created from the first source, namely that

\[\vec{u}^1(\vec{x}, t) = \vec{A}e^{i\omega_1(\vec{x} - \vec{x}_1)}, \quad (2)\]

where $\vec{A}$ will be represented by the asymptotic expansion, $\vec{A} = \vec{A}_0 + \vec{A}_1/(i\omega_1) + \vec{A}_2/(i\omega_1)^2 + \ldots$. Substituting this expansion into equation (1), writing the left side of equation (1) in powers of $\omega_1$ and setting the coefficient of the highest order terms of $\omega_1$ equal to zero results in, see (Ji and McLaughlin 2004),

\[0 = M\vec{A}_0, \quad (3)\]

where $M$ is the following matrix

\[M = [(\lambda + \mu)\nabla \phi_1(\nabla \phi_1)^T + (\mu|\nabla \phi_1|^2 - \rho)I] \quad (4)\]

The assumption here is that there is enough separation of scales so that the coefficient of each power of $\omega_1$ is separately equal to zero. For equation (3) to have a solution, the matrix $M$ must be singular. Setting the determinant of $M$ equal to zero yields that
either
\[ |\nabla \phi_1(\vec{x})| = \sqrt{\rho/\mu} = 1/C_s \]  \hspace{1cm} (5)

or
\[ |\nabla \phi_1(\vec{x})| = \sqrt{\rho/(\lambda + 2\mu)} = 1/C_p, \]  \hspace{1cm} (6)

where \( C_s \) and \( C_p \) are the shear and compression wave speeds respectively. In soft tissue, \( \lambda \) is several orders of magnitude greater than \( \mu \), (Sarvazyan et al. 1995). For the constant coefficient case in an elastic half space, the exact solution has been found in (Miller and Pursey 1954), and from this solution it is clear that the amplitude of the compression wave is very small, \( O((\mu/\lambda)^2) \), when the ratio \( \lambda/\mu \) is large. For this reason we will assume that equation (5) is satisfied. Likewise, we will write the complexification of the displacement, \( \vec{u}^2 \), from the second source as
\[ \vec{u}^2(\vec{x}, t) = \vec{B}e^{i\omega_2(-\phi_2(\vec{x}) - t)} \]  \hspace{1cm} (7)

and as above, the phase, \( \phi_2 \), satisfies
\[ |\nabla \phi_2(\vec{x})| = \sqrt{\rho/\mu} = 1/C_s \]  \hspace{1cm} (8)

Now, from our arguments in Section II, the spectral variance data from the crawling wave experiment is proportional to
\[ |u_r|^2 = |u_r^1 + u_r^2|^2 = A_r^2 + B_r^2 + 2A_rB_r\cos(\psi(\vec{x}, t)), \]  \hspace{1cm} (9)

where \( u_r = \vec{u} \cdot \hat{r} \), \( u_r^1 = \vec{u} \cdot \hat{r} \), \( u_r^2 = \vec{u}_2 \cdot \hat{r} \), \( \hat{r} \) is the unit radial variable and \( \psi(\vec{x}, t) = \omega_1(\phi_1(\vec{x}) - t) - \omega_2(-\phi_2(\vec{x}) - t) = (\omega_2 - \omega_1)t + \hat{\psi}(\vec{x}) \). So while the equations for \( \phi_1 \) and \( \phi_2 \) are interesting what we really want is an equation for \( \psi \). For the holographic wave experiment, the argument is similar except that \( B_re^{i\omega_2(-\phi_2(\vec{x}) - t)} \) is replaced by \( -B_re^{-i\omega_2t} \) and in the above equation for \( \psi \) we just have to set \( \phi_2 = 0 \).

Before deriving the Eikonal equation for \( \psi \) for either experiment we remark that the surfaces of constant phase, \( \psi(\vec{x}, t) = \psi_0 \), of the vibration amplitude are observed at a
fixed time as stripes in the moving interference pattern, see Figure 2 A, B for a snapshot of the moving interference pattern for the holographic wave and corresponding lines of constant phase. Furthermore, on any level surface, \( \psi(\vec{x}, t) = \psi_0 \), it can be shown, (see for example (Sethian 1999), Osher and Fedkiw 2002), and Appendix A in this paper), that the Eikonal equation, \(|\nabla \psi(\vec{x}, t)| F(\vec{x}) = \psi_1\), is satisfied, where \( F(\vec{x}) \) is the component of the velocity of the moving interference pattern, in the direction, \(-\nabla \psi\), which is normal to the level curve, \( \{ \vec{y} \mid \psi(\vec{y}, t) = \psi_0 \} \), at the point \( \vec{y} = \vec{x} \). In addition, because \( \psi_1 \) is the constant \( \omega_2 - \omega_1 \), only the spatially varying component of the phase, \( \hat{\psi}(\vec{x}) = \omega_1 \phi_1 + \omega_2 \phi_2 \), is present in the formula for the speed, \( F \), and so

\[
F = \frac{\omega_2 - \omega_1}{|\nabla \hat{\psi}(\vec{x})|} = \frac{\Delta \omega}{|\nabla \hat{\psi}(\vec{x})|}.
\]  

(10)

We use this equation and the arrival time algorithm to find \( F \) and use \( \omega_1 F/\Delta \omega \) and \( 2\omega_1 F/\Delta \omega \) as our imaging functionals for the the holographic wave and crawling wave experiments, respectively.

The speed \( F \) is not always a simple multiple of the shear wave speed. To show this we first calculate \(|\nabla \hat{\psi}(\vec{x}, t)|^2\) as

\[
|\nabla \hat{\psi}(\vec{x}, t)|^2 = |\omega_1 \nabla \phi_1(\vec{x}) + \omega_2 \nabla \phi_2(\vec{x})|^2
\]

\[
= \omega_1^2 |\nabla \phi_1(\vec{x})|^2 + \omega_2^2 |\nabla \phi_2(\vec{x})|^2 + 2\omega_1 \omega_2 \nabla \phi_1(\vec{x}) \cdot \nabla \phi_2(\vec{x})
\]

\[
= \omega_1^2 |\nabla \phi_1(\vec{x})|^2 + \omega_2^2 |\nabla \phi_2(\vec{x})|^2 + 2\omega_1 \omega_2 |\nabla \phi_1(\vec{x})||\nabla \phi_2(\vec{x})| \cos(\theta)
\]  

(11)
where $\theta$ is the angle between $\nabla \phi_1(\vec{x})$ and $\nabla \phi_2(\vec{x})$. Now, for the holographic wave experiment $\phi_2 = 0$ and

$$\frac{\Delta \omega^2}{|\nabla \psi|^2} = \frac{\psi_2^2(\vec{x}, t)}{|\nabla \psi(\vec{x}, t)|^2} = \frac{\Delta \omega^2 C_s^2}{(\omega_1^2)},$$

(12)

where $\Delta \omega = \omega_2 - \omega_1$. So for this case

$$F = \frac{\Delta \omega C_s}{\omega_1}.$$  

(13)

For the crawling wave experiment the relationship between the crawling wave speed, $F$, and the shear wave speed, $C_s$, is more complicated. There using equations (5), (8), and (11) we have

$$|\nabla \hat{\psi}(\vec{x}, t)|^2 = (1/C_s^2)(\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos(\theta)).$$

(14)

Again, letting $\omega_2 = \omega_1 + \Delta \omega$ we calculate the ratio $(\omega_2 - \omega_1)/|\nabla \psi(\vec{x}, t)|^2$ as

$$F^2 = \frac{\Delta \omega^2}{|\nabla \hat{\psi}(\vec{x})|^2} = \frac{\psi_1^2}{|\nabla \psi|^2} = \frac{\Delta \omega^2 C_s^2}{2\omega_1^2(1 + \cos(\theta)) + O(\omega_1\Delta \omega_1) + O(\Delta \omega^2)}$$

(15)

$$= \frac{\Delta \omega^2 C_s^2}{4\omega_1^2 \cos^2(\theta/2) + O(\omega_1\Delta \omega_1) + O(\Delta \omega^2)}.$$  

(16)

This equation can’t be used to directly find the wave speed, $C_s(\vec{x})$, from the phase, $\hat{\psi}(\vec{x}, t)$, because the $\cos(\theta)$ term depends on the unknowns $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$. However, equations (5), (8), and (15) are a coupled system of three equations that can be solved for $C_s$, $\phi_1(\vec{x})$, and $\phi_2(\vec{x})$. This will be the subject of a future paper. So here for the crawling wave experiment we will only consider using the quantity $2\omega_1/|\nabla \hat{\psi}(\vec{x})|$ as an imaging functional.

**Remark:** Here we would like to make a few more remarks. The first is that we have only one component of the vibrational amplitude. While all components of the vibrational amplitude have the same phase under the assumption that a geometric optics expansion is valid, this component must have significant enough amplitude to find the phase, $\hat{\psi}$. Furthermore, the vibration amplitude is measured in a plane. So, the out of plane component of the gradient of the phase can’t be calculated in equation (15).
and we will assume it is zero. However, if there is significant out of plane motion of
the moving interference pattern our assumption will cause overestimation of the speed.
There is one other distinguishing feature for the holographic experiment. The direction
of propagation of the shear wave is $\nabla \hat{\psi} = \omega_1 \nabla \phi_1$ so the moving interference pattern
moves either in the same (if $\omega_2 > \omega_1$) or directly opposite direction (if $\omega_2 < \omega_1$) as the
shear wave induced by the source vibrating in the phantom (or tissue). For the crawling
wave experiment, $\nabla \hat{\psi} = \omega_1 \nabla \phi_1 + \omega_2 \nabla \phi_2$ so the direction of the moving interference
pattern is not, in general, in the same direction as either of the shear waves induced
individually by the two sources.

V. Calculating phase and arrival time for the crawling and holographic
wave experiments

As described above, given the phase, $\hat{\psi}$, the speed, $F$, of a moving interference pattern
can be obtained from the Eikonal equation, $|\nabla \hat{\psi}|F = \Delta \omega$. To utilize this equation we
must first construct the phase from the data $|u_r|^2 = A_r^2 + B_r^2 + 2A_r B_r \cos (\Delta \omega t + \hat{\psi})$.
This is related to the classic phase unwrapping problem. Here our goal is to construct
a unique $\hat{\psi}$ which varies continuously with $\vec{x}$ as our equation for $F$ contains derivatives
of $\hat{\psi}$.

We can interpret a multiple of $\hat{\psi}$ as an arrival time. This is based on the
following observation. At an arbitrary fixed point, $x_0$, the time trace of the vibration
amplitude, $|u_r(x_0, t)|^2$, can be represented by $M_0 + 2(A_r B_r)(x_0) \cos((\omega_2 - \omega_1)t + \hat{\psi}(x_0))$,
where $M_0$ is an unknown constant. Now remove the additive constant $M_0$ from the
signal, (say by filtering out the zero frequency in frequency space), and consider the
remaining signal with $M_0$ removed, $V(x_0, t) = 2(A_r B_r(x_0)) \cos((\omega_2 - \omega_1)t + \hat{\psi}(x_0))$;
an example of this is shown in Figure 3 A with a blue line. Now consider the
time trace at a second fixed point, $x_1$, with the additive constant also removed,
Figure 3. A Reference signal $S_1 = \cos((\omega_2 - \omega_1)t)$ (blue line) and a time trace of the vibration amplitude, $v(\tilde{x}_0, t)$ (red line) B Cross-correlation function of the two signals $S_1$ and $v(\tilde{x}_0, t)$.

$$V(\tilde{x}_1, t) = 2(A_r B_r)(\tilde{x}_1)(\cos((\omega_2 - \omega_1)t + \hat{\psi}(\tilde{x}_1))).$$

Notice that, except for magnitude, $2(A_r B_r) V(\tilde{x}_1, t)$ is very nearly $V(\tilde{x}_0, t)$ except that it is time delayed by $\hat{\psi}(\tilde{x}_1) - \hat{\psi}(\tilde{x}_2)$ seconds. That is $(\omega_2 - \omega_1)t + \hat{\psi}(\tilde{x}_0) = (\omega_2 - \omega_1)(t - \delta)t + \hat{\psi}(\tilde{x}_1)$, and we can interpret $\hat{\psi}(\tilde{x}_1) - \hat{\psi}(\tilde{x}_0)$ as a scaled time delay. So it is appropriate to define the quantity $\hat{\psi}(\tilde{x}_1)/\Delta \omega = T(\tilde{x}_1)$ as an arrival time, $T(\tilde{x}_1)$, of the signal $V(\tilde{x}_0, t)$ at the point $\tilde{x}_1$.

With this in mind, for the rest of this paper, we will refer to the scaled phase $\hat{\psi}(\tilde{x})/\Delta \omega$ as the arrival times, $T(\tilde{x})$.

We compute the arrival time using:

$$C(\tilde{x}, \delta t) := \frac{1}{T} \int_0^T \tilde{v}(\tilde{x}_0, t)\tilde{v}(\tilde{x}, t - \delta t) dt,$$

where

$$\tilde{v}(\tilde{x}, t) = \begin{cases} 
V(\tilde{x}, t) & \text{if } 0 \leq t \leq \hat{T}, \\
V(\tilde{x}, t - \hat{T}) & \text{if } t > \hat{T}, \\
V(\tilde{x}, t + \hat{T}) & \text{if } t < 0.
\end{cases}$$
Now we estimate the arrival times by $T(\vec{x}) \approx \delta t_{\text{max}}$, where

$$\delta t_{\text{max}} := \arg \max_{\delta t \in [0, T]} C(\vec{x}, \delta t).$$

That is $T(\vec{x})$ is estimated by the time delay, $\delta t$, that maximizes the correlation between the signals $\tilde{v}(\vec{x}_0, t)$ and $\tilde{v}(\vec{x}, t - \delta t)$. Of course, since the two signals are cyclical, there are many local maxims of nearly the same amplitude. An example cross-correlation function is shown in Figure 3 B. To handle this inherent non uniqueness, we choose one of the maxims arbitrarily for the first point, $\vec{x}_0$. Since the cross-correlation function at the point, $\vec{x}_0$, is the auto-correlation function, we choose $T(\vec{x}_0) = 0$. Then, we look for local maxima in the cross correlation function at the neighboring points of $\vec{x}_0$ near the value $T(\vec{x}_0)$. To add some stability to this procedure, when finding the arrival time, $T(\vec{x})$, at a new point, $\vec{x}$, we use the median value of $T$ at nearby points that already have a computed arrival time, as a starting point.

We will use these computed arrival times as input to an inverse Eikonal solver described below, see also (Ji et al. 2003a, McLaughlin and Renzi 2006a, 2006b). The output of this solver will be the speed of the moving interference pattern.

VI. Solving the Inverse Eikonal Equation

Here we describe how to find the interference pattern speed, $F$, where $2\omega_1 F/\Delta \omega$, and $\omega_1 F/\Delta \omega$ are our imaging functionals for the crawling and holographic wave experiments respectively. The goal now is to calculate $F = |\nabla T|^{-1}$ in a smart way, avoiding the essentially unstable calculation of dividing by derivatives of noisy data.

VI.1. Distance and level curve methods for the Arrival Time Algorithm

A slow but robust, second order method approximates the speed of the moving interference pattern using the elementary idea that speed is distance divided by time.
So

\[
F \approx \left\{ \frac{1}{2\Delta t} (\min_{\tilde{x}^+} |\tilde{x} - \tilde{x}^+| + \min_{\tilde{x}^-} |\tilde{x} - \tilde{x}^-|) : \tilde{x}^\pm \text{satisfies } T(\tilde{x}^\pm) = T(\tilde{x}) \pm \Delta t \right\}. \quad (17)
\]

We call this method for finding \( F \) the distance method. This is justified in (McLaughlin and Renzi 2006a). A faster \( O(m \log m) \) algorithm is described below.

Starting with the surface \( S_T = \{ (\tilde{x}, t) \mid T(\tilde{x}) = t, 0 < t < T, \tilde{x} \in \Omega \} \) where \( \Omega \) is the image plane, define the higher dimensional function \( \gamma(\tilde{x}, t) = \pm \min\{|\tilde{x} - \tilde{x}^\pm| : \tilde{x} \text{satisfies } T(\tilde{x}) = t\} \) where plus (minus) is chosen if \( t > T(\tilde{x}) \) \( (t < T(\tilde{x})) \), respectively. Then

\[
\gamma(\tilde{x}, T(\tilde{x})) = 0 \text{ for } \tilde{x} \in \Omega \text{ and } |\nabla \gamma| = 1 \text{ so that }
\]

\[
\gamma_t = |\nabla T|^{-1} = F \text{ on } \{(\tilde{x}, t) \mid \gamma(\tilde{x}, t) = 0\} = S_T.
\]

The potentially unstable term \( |\nabla T|^{-1} \) is now replaced by \( \gamma_t \) which is in the numerator and furthermore no further approximations are made to achieve this equation (see Osher and Fedkiw (2002), Sethian (1999), and Appendix A). For our inverse problem to obtain the \( O(m \log m) \) algorithm speed, the extension from \( S_T \) to \( \gamma \) is made quickly and simultaneously for all times in our discretization. For those details we refer to (McLaughlin and Renzi 2006b). The full algorithm for calculating the speed, \( F \), in this way is called the level curve algorithm. Note that as a final step we apply total variation minimization, (Rudin et al. 1992).

**VII. Phantom Experiments**

Combining the ideas of Section V (arrival time calculation ) and Section VI (speed calculation from arrival times) gives a complete algorithm to recover interference pattern speed. We test this algorithm on data obtained from experiments performed at the Center for Ultrasound at the University of Rochester. For these experiments, the Zerdine tissue mimicking phantom (CIRS Norfolk, Va) is bowl-shaped and approximately 15
Figure 4. A Snapshot of the moving interference pattern in the crawling wave experiment; B Imaging functional, $2\omega F/\Delta \omega$, related to the shear wave speed in the crawling wave experiment. The units for the axis are millimeters and the color bar units are $m/s$.

× 15 × 15 cm in size. The phantom contains an isotropic background and a 1.3 cm diameter spherical stiff inclusion. The shear wave speed in the stiff inclusion is approximately $\sqrt{7} \approx 2.65$ times faster than the background shear wave speed.

We first present the results of our algorithm applied to data from the moving interference pattern that occurs in the crawling wave experiment. In this experiment, two vibration sources are used on opposite ends of the phantom at slightly different frequencies, 250 Hz and 250.15 Hz. Figure 4 shows a snapshot of the interference pattern in a middle region in the plane containing the two sources and the ultrasound transducer.

The first step to generate a shear wave speed reconstruction is to find the arrival times, $T$, from the spectral variance data. Before we do this, we preprocess the data. We use the 1D fast Fourier transform on each time trace, and filter out all the frequencies except for a narrow band around the driving frequency, $\Delta \omega = .15 Hz$. Then we find the phase as discussed in Section VI. The interference pattern speed, $F = |\nabla T|^{-1}$, is calculated with the inverse level curve method for the Arrival Time Algorithm. The imaging functional, $2\omega_1 F/\Delta \omega$, is shown in Figure 4 B. The wave speed contrast of the reconstruction is about 2.33, which is very close to the actual wave speed contrast of
2.65. Note also the ring like artifact around the recovered inclusion. This is likely due to the omission of the \( \cos(\theta/2) \) term in our equation for the speed.

The astute reader may wonder why the interference patterns look very similar to a plane wave, when two point sources are used. At the end of Section IV, we remarked that the interference pattern may not move in the direction of either of the two propagating shear waves. As we described before, at fixed time, the gradient of the phase of the vibration amplitude, \( \nabla \hat{\psi} \), is up to a multiplicative constant, nearly the sum of the gradients of the phases of the two shear waves, \( \nabla \phi_1 + \nabla \phi_2 \). The propagation direction is determined by the gradient of the phase. Considering the term \( \nabla \hat{\psi} \approx \nabla \phi_1 + \nabla \phi_2 \), see (9), that we derived using the crawling wave setup we have: (1) the two point sources are placed on opposite ends of the medium; and (2) the imaging plane is in the middle region between the sources. In this case the vertical components of \( \nabla \phi_1 \) and \( \nabla \phi_2 \) will have opposite sign in the phantom and so heuristically the vertical components of the sum, \( \nabla \phi_1 + \nabla \phi_2 \), will nearly cancel and the horizontal components will add. To demonstrate we solve the equations \( |\nabla \phi_1| = 1 \), and \(-|\nabla \phi_2| = -1\), on a 15 cm X 15 cm square with the initial conditions \( \phi_1(0, 7.5) = 0 \), and \( \phi_2(15, 7.5) = 0 \). The lines of constant phase for \( \phi_1 \) and \( \phi_2 \) are simply expanding circles and are shown in Figure 5 A, B. For \( \phi_1 + \phi_2 \) the lines of constant phase, shown in Figure 5 C, are quite a bit more complicated. However, near the line equidistant to the two point sources the lines of constant phase for \( \phi_1 + \phi_2 \) are nearly vertical. This elementary demonstration motivates our statement that the lines of constant phase for \( \phi_1 + \phi_2 \) are similar to those from a plane wave source (Figure 5 F). For comparison, we again show the lines of constant phase for \( \phi_1 \) and \( \phi_2 \) in Figure 5 D, E. Also, note that there are twice as many contours in Figures 5 C, F as there are in Figures 5 A, B, D, E. This is the reason there is a factor of 2 in the formula for the crawling wave imaging functional.

Next we present the results of our algorithm applied to data from the holographic
Figure 5. A Lines of constant phase for $\phi_1$; B Lines of constant phase for $\phi_2$; C Lines of constant phase for $\phi_1 + \phi_2$; D lines of constant phase for $\phi_1$ on a small middle region; E Lines of constant phase for $\phi_2$ on a small middle region; F Lines of constant phase for $\phi_1 + \phi_2$ on a small middle region.

Figure 6. A Snapshot of the moving interference pattern in the holographic wave experiment; B Recovery of the shear wave speed $C_s$ in the holographic wave experiment. The units for the axis are millimeters and the color bar units are m/s.
wave experiment. In this experiment, one vibration source at 200 Hz is used to generate a shear wave. The ultrasound transducer is also vibrated at a slightly different frequency, 200.1 Hz. Figure 6 A shows a snapshot of the moving interference pattern. Note that, as explained at the end of Section IV, the moving interference pattern should propagate in the same direction as the underlying shear wave. Here we observe what looks like an expanding half-circle as one would expect from a point source. We find the arrival times and the interference pattern speed, $F$, as outlined in Sections V and VI. In this case, the shear wave speed can be calculated as $C_s = \omega_1 F / \Delta \omega$ and is shown in Figure 6 B. Note that the imaging plane in the two experiments are at slightly different locations in the phantom. The black circle indicates the size of the stiff inclusion. In this reconstruction the wave speed contrast is almost 2, compared to the actual wave speed contrast of 2.65. Note that there are fewer artifacts in this reconstruction. This is due to the more accurate relationship between the interference speed and the shear wave speed for the holographic wave experiment.

VIII. Conclusion

We have developed a new algorithm to image the speed of moving interference patterns. This algorithm is composed of two sub-algorithms. The first sub-algorithm finds the arrival times of one of the stripes of the moving interference pattern. The second sub-algorithm takes as input the arrival times found by the first sub-algorithm, and finds the moving interference speed by solving the inverse Eikonal equation using the inverse level curve method for the Arrival Time Algorithm. With data from a crawling wave experiment, a multiple of the speed of the moving interference pattern is used as an imaging functional for the shear stiffness because the relationship between the moving interference pattern speed and the shear stiffness is quite complicated. Note also that this image has more artifacts than the one obtained from holographic wave
data. We expect fewer artifacts when in future work we consider equations (5), (8), and 
\[ 1/|\nabla T| = \frac{\Delta \omega C_s}{2\omega_1 \cos(\theta/2)} = F \]
all together. For the holographic wave experiment there is a simple linear relationship between the moving interference speed and the shear wave speed, so in that case we image the shear wave speed and obtain fewer artifacts. For both experiments there are artifacts due to the low bit rate achieved with only a 16 color differentiation in the display. Note, also, that with a single source the amplitude of the wave decreases as we move away from the source so that in human tissue the amplitude loss may be too great to obtain good images at some distance from the source. If this is the case, the additional work of solving (5), (8), and 
\[ 1/|\nabla T| = \frac{\Delta \omega C_s}{2\omega_1 \cos(\theta/2)} \]
will be essential.

Here, we have tested our algorithm with data obtained by Zhe Wu in the laboratory of Kevin Parker, in the Center for Biomedical Ultrasound at the University of Rochester. They performed both the one and two frequency experiments using a tissue mimicking Zerdine phantom containing a 1.3 cm diameter circular inclusion. We have obtained very good shear stiffness images with both sets of experimental data.

IX. Appendix A

In this appendix we show that, on a level surface of the phase, \( \psi(\vec{x}, t) = k \); the equation
\[ \psi_t = |\nabla \psi| F \] (18)
is satisfied where \( F \) is the component of the velocity in the direction, \( -\nabla \psi \), which is normal to the level curve \( \psi(\vec{x}, t) = k \), fixed \( k \). Let \( \vec{X}(t) \) be a parametric representation of a point lying on \( \psi(\vec{x}, t) = k \) with \( x(t_0) = x_0 \), some \( (\vec{x}_0, t_0) \) satisfying \( \psi(\vec{x}_0, t_0) = k \). Since \( -\nabla \psi/|\nabla \psi| \) is normal to the curve \( \psi(\vec{x}, t_0) = k \), at \( \vec{x} = \vec{x}_0 \),
\[ F(\vec{X}) = \vec{X}_t \cdot ( -\nabla \psi / |\nabla \psi| ) |\nabla \psi| . \] (19)
Taking a time derivative of \( \psi(\vec{x}(t), t) = k \) yields
\[ \psi_t + \nabla \psi \cdot \vec{X}_t = 0. \] (20)
Shear stiffness recovery using interference patterns

Multiplying through the above equation by $1/|\nabla \psi|$, and using equation (19), leads to

$$\psi_t = |\nabla \psi|F. \tag{21}$$

Since $(\bar{x}_0, t_0)$ are arbitrarily chosen on $\psi(\bar{x}, t) = k$, formula (18) is established.


Shear stiffness recovery using interference patterns


Shear stiffness recovery using interference patterns


**List of Figures**

(i) Experimental setup ...............................................................................................7

(ii) Snapshot of a moving interference pattern ............................................................12

(iii) Cross-correlation function ....................................................................................15

(iv) Lines of constant phase ........................................................................................18

(v) Crawling wave speed recovery ..............................................................................20

(vi) Holographic wave speed recovery .......................................................................20