Name: ____________________________

SS No.: ____________________________

INSTRUCTIONS

1. Please put your name on this cover sheet and then wait for the announced start of the exam before reading the questions.

2. The exam is open book and open notes.

3. Use the indicated space for your answers to the exam questions and any additional material that you wish to have graded.

4. Promptly turn in this exam booklet intact at the announced end of the exam.

5. Pertinent rules of the University board on Academic Honesty apply to the conduct of this exam.

6. Please note that the problems are not equally weighted.
Problem 1. (15%)  

Consider the random process $X(t)$ defined as  

$$X(t) = A e^{j(Wt + \theta)}$$  

where $A, W,$ and $\theta$ are statistically independent random variables. The random variables have the following statistics:  

- $A$ has mean $\eta$ and variance $\sigma^2$,  
- $\theta$ is uniform on $[0, 2\pi]$,  
- $W$ is uniform on $[\omega_1, \omega_1 + \Delta\omega]$  

Determine the autocorrelation function $R_X(\tau)$ for $X(t)$. 
Problem 2. (15%) 

a) (8%) Show that \( \phi(\omega) \) given below is a characteristic function.

\[
\phi(\omega) = \left( \frac{1}{2} e^{j\omega} + \frac{1}{2} \right)^3 \left[ e^{j\omega} - 1 + j\omega - \omega^2 \right]
\]
b) (7%) Determine whether or not $\phi(\omega)$ below is a characteristic function.

$$\phi(\omega) = e^{-\omega^4}$$
Problem 3. (15%)

A WSS random process, $X(t)$, has autocorrelation function $R_X(\tau)$ given by

$$R_X(\tau) = 100e^{-10|\tau|} + 50e^{-20|\tau|} + 100$$

a) (8%) Determine the mean, $\mu$, and the variance, $\sigma^2$, of the process.
b) (7%) Find the **Power Spectral Density** of the process, \( S_X(\omega) \).
Problem 4. (15%)  
The input of a linear system characterized by impulse response $h(t)$ is a WSS random process, $X(t)$, with ACF given by 

$$R_X(\tau) = \frac{N_0}{2} \delta(\tau)$$

Calculate the variance of the output process $Y(t)$ in terms of $H(j\omega)$, the Fourier transform of the impulse response.
Problem 5. (15%)

A linear time-invariant system has impulse response \(h(t)\) given by

\[
h(t) = \delta(t) + ke^{-at} u(t)
\]

Determine a relationship between \(a\) and \(k\) such that the input and output WSS random processes have the same autocorrelation function.
Problem 6. (25%)
A random process, \( X(t) \), is composed of a deterministic periodic signal, \( s(t) \), plus additive white noise, \( N(t) \). Thus

\[
X(t) = s(t) + N(t)
\]

where \( s(t + T) = s(t) \), for all \( t \).

It is suggested that if the signal is **periodically added**, as discussed below, that the signal will grow and the noise will tend to cancel out. We will periodically add the data process by forming the sum process, \( X_n(t) \), as

\[
X_n(t) = \frac{X(t) + X(t + T) + X(t + 2T) + \cdots + X(t + nT)}{n + 1}
\]

The noise, \( N(t) \), is WSS with zero mean and PSD which is constant, \( N_0 \) for all frequencies.

a) (15%) Calculate the mean and variance of the sum process.
b) (10%) Show that this technique can completely remove the noise by discussing the properties of the sum process as $n \to \infty$. You may use anything learned in this class or any other class to discuss this point.
Consider the differential equation
\[ a y(t) + b g(t) = 0, \quad y(0) = y_0, \quad y(t) = 0, \quad t > 0. \]

We allow \( a \) and \( b \) to be random variables independent of \( t \). One can see that \( y(t) \) is in general a random process.

a) Solve for \( y(t) \) in terms of \( a, b, y_0 \).

b) Let
\[ f_A(a) = f_B(b) = \begin{cases} 2, & 0 < a < 1 \\ 0, & \text{elsewhere} \end{cases} \]

Determine \( \mu_y(t) \) and \( \gamma_y(t; \tau) \), the mean and autocorrelation function of \( y(t) \).

2. Consider the random process \( X(t) \), defined as follows:

\[ X(t) = \ldots, -a_2, a_1, a_0, a_1, \ldots \]

where \( a_2, a_1, a_0, \ldots \) are a sequence of independent, identically distributed random variables with \( E[a_i] = m \), \( \text{Var}[a_i] = \tau^2 \).

The point \( t_0 \) can be any value in \(( -\frac{T}{2}, \frac{T}{2} )\) with uniform probability. This allows us to say that the process has uniformly distributed phase.
2] continued

a) Is \( x(t) \) a wide sense stationary process? Justify your conclusion.

b) Calculate \( M_x(t) = R_x(t_2, t_2) \) for the process.

c) Discuss what changes would take place for \( \Delta > T/2 \).

Consider \( x(t) \) defined as:

\[
\begin{array}{c}
\vdots \\
\cdots
\end{array}
\]

\( x(t) \) may with equal probability assume at any instant of time either of the values zero or one, and it makes independent random traversals from one value to another. We assume that \( k \) traversals occurring in \( T \) seconds have probability

\[
P(k; T) = \left( \lambda T \right)^k \frac{e^{-\lambda T}}{k!},
\]

where \( \lambda \) is the average number of traversals per unit time.

Calculate \( R_x(t) \) for the process. (Show details).

1] For the process in Problem 2]. Calculate:

\[
\hat{M}_T = \frac{1}{2T} \int_{-T}^{T} x(t) \, dt \quad \text{and} \quad \hat{R}_T(t) = \frac{1}{2T} \int_{-T}^{T} x(t+t_2) x(t) \, dt.
\]

and determine \( \lim_{T \to \infty} \hat{M}_T \), \( \lim_{T \to \infty} \hat{R}_T(t) \). What conclusion do you arrive at?
Problem 1

(i) Assuming \( y(t) = ke^{-mt} \), \( k, m \) constants, we obtain

\[
y(t) = \begin{cases} \frac{b}{a} - ct \quad & t < 0 \\ 0 & t \geq 0 \end{cases}
\]

N.B. \( E[y(t_1)Y(t_2)] = y_0 E[y(t_1 + t_2)] = y_0 M_X(t_1, t_2) \)

So we need only calculate \( M_X(t) = \) 

To do so we must have \( f_C(z) \)

\[
f_C(z) = \int_{-\infty}^{\infty} |x| f_C(z,x) f_a(x) dx \quad \text{(Pairwise)}
\]

\[
= \begin{cases} \int_0^z x dx, & 0 \leq z \leq 1 = \frac{z}{2} \\ \frac{z^2}{2}, & 1 < z \leq 2c^2 \\ 0, & z > 2c^2 
\end{cases}
\]

\[
E[y(t)] = y_0 E[e^{-ct}] = y_0 \int_{0}^{\infty} e^{-ct} f_C(z) dz
\]
\[ E[Y(t)] = \frac{y_0}{2} \int_0^t e^{-ct} \, dc + \frac{y_0}{2} \int_t^\infty \frac{e^{-ct}}{c^2} \, dc \]

\[ \theta = \frac{y_0}{2} \left[ \frac{e^{-ct}}{-t} \right]_0^t = -\frac{y_0}{2t} \left[ e^{-c} - 1 \right] = \frac{(1-e^{-t})y_0}{t} \]

\[ \phi = \frac{y_0}{2} \int_t^\infty \frac{e^{-ct}}{c^2} \, dc = \frac{y_0}{2} \left[ \frac{-e^{-ct}}{c} \right]_t^\infty + t \int_t^\infty \frac{e^{-ct}}{c} \, dc \]

\[ = \frac{y_0}{2} \left[ e^{-t} + t \left( -1 + \int_t^\infty \frac{e^{-p}}{p/t} \, dp \right) \right] \quad \text{at } t = p \quad dc = \frac{dp}{t} \]

\[ \Delta + \phi = m_Y(t) = \frac{y_0}{2} \left[ \frac{1 - e^{-t}}{t} + e^{-t} + t E_x(-t) \right] \]

\[ R_Y(t_1, t_2) = \frac{y_0^2}{2} \left[ \frac{1 - e^{-t_1}}{t_1} + e^{-t_1} + t_2 E_x(-t_2) \right] \]
Problem 2: a) + b) Consider a time (e.g., $t_i$) and a typical ensemble member. Let $t_i \in (t_0, t_0 + T)$.

Because $t_0$ is uniform over $(-T/2, T/2)$, there is no point $t$, so that the pulse could not be hit. If, however, we were to change the region of $t_0$, we would get something different.

Let $t_1$ be uniform over $(-T/4, T/4)$, and again let $t_1$ be anywhere in $(t_0, t_0 + T)$. There would be a region $\{ t_1 \in (t_0 + T/8, t_0 + 7T/8) \}$ for which a pulse would never hit. Thus the distribution of the process would certainly depend upon whether or not we were inside or outside this interval. Hence the process would not be stationary.

Now with $t_0 \in (-T/2, T/2)$, any ensemble member can start at any point in $(-T/2, T/2)$, and therefore the probability of hitting or missing the pulse is independent of time (stationary).

$$P[\text{hitting pulse}] = \frac{1}{2}, \quad P[\text{miss}] = 1 - \frac{1}{2}, \quad \forall t_i.$$
To calculate $m_x$:

\[ m_x = E[x(t)] = \frac{\Delta}{T} E[a_i] + (1 - \frac{\Delta}{T}) E[0] = \frac{\Delta}{T} m \]

Now we must calculate the auto-correlation function: $R_x(\tau)$

$\tau = 0$ \quad $E[x(t)]:$ \quad we calculate $m_x^2$:

\[ m_x^2 = \frac{\Delta}{T} E[a_i^2] + (1 - \frac{\Delta}{T}) E[0] = \frac{\Delta}{T} [r^2 + m^2] \]

If $\tau = nT$

\[ E[x(t)x(t+nT)] = \frac{\Delta}{T} E[a_i a_{i+1}] = \frac{\Delta}{T} E^2[a] = \frac{\Delta}{T} m^2 \]

$0 \leq |\tau| < \Delta$
Now the probability of a hit has been reduced by the fact that product $x(t)x(t+2)$ is zero for more $t$ in $(t_0, x_0 + 2)$.

\[ \mathbb{P}[\text{hit}] = \frac{\Delta - |x_1|}{T} \begin{cases} \{x_1, \text{ because } z \text{ can be} \\ + or - \text{ with same result} \} \end{cases} \]

By symmetry, we have:

\[ E[x(t)x(t+2)] = \frac{\Delta - |x_1|}{T} \mathbb{E}[x_3^2] = \frac{\Delta - |x_1|}{T} \left( \frac{s^2 + m^2}{2} \right) \]

\[ 0 \leq |x_1| \leq \Delta \]

Since $x(t)$ and $x(t+2)$ are disjoint in

\[ 0 \leq |x_1| \leq \Delta, \quad R_x(z) = 0 \quad \text{for this } z \]

Also, for $\Delta \leq |x - nT| \leq \Delta$, $R_x(z) = 0$.

Combining these results yields:

\[ R_x(z) = \begin{cases} \left( \frac{\Delta}{T} - \frac{|x_1|}{T} \right) \left( \frac{s^2 + m^2}{2} \right), & 0 \leq |x_1| \leq \Delta \\ 0, & 0 \leq |x - nT| \leq \Delta \\ \left( \frac{\Delta}{T} - \frac{|x - nT|}{T} \right) m^2, & 0 \leq |x - nT| \leq \Delta, \quad n \neq \Delta, \Delta \end{cases} \]
Problem 3] done in D+R P. 61

(Its an reserve)
c) For φ = 0, we see that the triangle must be drawn so that

\[ R(x) \] can be drawn for

\[ A = 0 \]
Problem 4.1

a) \[ \hat{M}_{T_0} = \frac{1}{2T_0} \int_{-T_0}^{T_0} x(t) \, dt \]

\[ 2T_0 \approx (2N+1)T \]

\[ \hat{M}_{T_0} \approx \frac{1}{(2N+1)T} \sum_{n=-N}^{N} a_n \Delta, \quad \text{area} = a_n \Delta \]

\[ \frac{\Delta}{T} \left[ \frac{1}{2N+1} \sum_{n=-N}^{N} a_n \right] \]

Now by WLLN: \[ \frac{1}{2N+1} \sum_{n=-N}^{N} a_n \approx E[a] = m \quad \text{as } T \to \infty \]

\[ \Rightarrow a \sim N \to \infty \]

\[ \frac{\Delta}{T} \hat{M}_{T_0} = \frac{\Delta}{T} m = \hat{m}_X \]

Thus \( x(t) \) is ergodic in the mean. Now we consider the autocorrelation
\[
\hat{R}_{T_0}^{(2)}(x) = \frac{1}{(2N+1)^T} \sum_{n=-N}^{N} a_n^2 \Delta \approx \frac{\Delta}{N} E[a^2] \text{ by WLLN}
\]

\[
\sum_{n=-N}^{N} a_n^2 \text{ is an unbiased estimator of } E[a^2]
\]

and \( \sum_{n=-N}^{N} a_n^2 \) are ind, ident. dist. R.V.'s.

\[
\lim_{T_0 \to \infty} \hat{R}_{T_0}^{(2)}(x) = \frac{\Delta}{N} E[a^2] = \frac{\Delta}{N} \left[ N^2 + m^2 \right]
\]

\[
\begin{align*}
R_{T_0}^{(N)}(nT) &= \frac{1}{(2N+1)^T} \sum_{n=-N}^{N} a_n a_{n+m} \Delta \\
&= \frac{1}{N} \sum_{n=-N}^{N} a_n a_{n+m} \Delta
\end{align*}
\]

\[
\text{Now } \frac{1}{2N+1} \sum_{n=-N}^{N} a_n a_{n+m} \text{ is an unbiased estimator of } E[a]
\]

\[
R_x^{(N)}(nT) = \frac{1}{N} \left[ m^2 \right]
\]
\[ 0 \leq |x| \leq D \]

\[ R_{T_0}(x) = \frac{1}{(2N+1)T} \sum_{n=-N}^{N} a_n^2 \left[ \bar{D} - |x| \right] \]

area of one box of \( x(t) \times (t+1) \)

\[ \therefore \lim_{T \to \infty} R_{T_0}(x) = \frac{D - |x|}{T} \left( c^2 + m^2 \right), \quad 0 \leq |x| \leq D \]

and so forth:

So that conclusion is \( x(t) \) is also ergodic in the autocorrelation.
Problem 1. Construct a two dimensional joint distribution for $(x, y)$ such that $x, y$ are statistically independent but $z = x + y$ and $w = x - y$ are not.

Problem 2. Determine whether or not the following function are characteristic functions:
   a) $\cos^3(\omega)(\frac{1}{2} + \frac{1}{2}e^{j\omega})^4 e^{2(e^{j\omega} - 1)}$
   b) $e^{-\omega^4}$

Problem 3. A continuous random variable, $X$, has p.d.f $F_X(x)$. We form a function of this random variable $y = g(x)$ where for the function $g(x)$ we choose $F_X(x)$. Determine the distribution function for $y$.

Problem 4. A linear system has impulse response $h(t) = e^{-t}u(t)$

100 such systems are cascaded to form $h_T(t) = [h * h * \cdots * h]_{100}(t)$

Since $h(t)$ is a density function, $h_T$ is also a density function.
a) Determine the mean and variance of the random variable whose density function is $h(x)$.

b) Determine the mean and variance of the random variable whose density function is $h_\tau(x)$.

c) Discuss the exact form for $h_\tau(x)$ and determine a useful approximate form using the central limit theorem.

Hint:
\[
\mathcal{L}\left[\frac{t^n}{n!}\right] = \int_0^\infty t^n e^{-st} dt = \frac{1}{s^{n+1}}
\]
5. Given a stationary process \( X(t) \) with \( R_x(t) \) continuous at the origin i.e.
\[
\lim_{\epsilon \to 0} |R_x(t \pm \epsilon) - R_x(0)| = 0
\]
show
\[
\lim_{\epsilon \to 0} |R_x(t \pm \epsilon) - R_x(0)| = 0
\]

Now
\[
|R_x(t \pm \epsilon) - R_x(0)|^2 = \left| E[X(t \pm \epsilon)X^*(t)] - E[X(t)X^*(t)] \right|^2
\]
\[
= \left| E[X(t \pm \epsilon)X^*(t)] - E[X(t)X^*(t)] \right|^2
\]
\[
\leq E \left[ \left( \|X(t \pm \epsilon) - X(t)\| \right) \|X^*(t)\| \right]^2
\]
by Schwarz's inequality
\[
\leq E[|X(t \pm \epsilon) - X(t)|^2]E[|X^*(t)|^2]
\]
\[
= E[|X(t \pm \epsilon) - X(t)|^2]E[|X^*(t)|^2]
\]
\[
= E[|X(t \pm \epsilon) - X^*(t)\|^2] + E[|X(t) - X^*(t)|^2] - E[|X(t \pm \epsilon) - X^*(t)|^2]
\]
\[
= \left\{ R_x(0) - R_x(\pm \epsilon) - R^*(\pm \epsilon) \right\} + \left\{ 2R_x(0) - R_x(\pm \epsilon) - R_x(0) \right\}
\]
\[
= \left\{ 2R_x(0) - R_x(\pm \epsilon) - R_x(0) \right\}
\]
Now since $X(t)$ is a stationary process and we assume $P[X(t) \neq 0] > 0$, $P_x(0) > 0$.

Now
\[
\lim_{\varepsilon \to 0} |P_x(t + \varepsilon) - P_x(t)|^2 \leq \lim_{\varepsilon \to 0} \left[ |2P_x(0) - P_x(\varepsilon)| \cdot 2P_x(0) \right] = 0
\]

Since
\[
\lim_{\varepsilon \to 0} |P_x(t + \varepsilon) - P_x(t)| = 0
\]

And finally
\[
\lim_{\varepsilon \to 0} |P_x(t + \varepsilon) - P_x(t)| = 0
\]

\[\square\]
4. Cesaro $R_x(2)$

\[ R_x(2) = \sigma^2 \Delta \left( 1 - \frac{21}{2\Delta} \right) + \sum_{n=-\infty}^{\infty} \frac{\Delta}{2\Delta} \left( 1 - \frac{|2nT+\Delta|}{\Delta} \right) \]

\[ \begin{cases} \frac{\Delta}{2} & \text{if } nT - \Delta < 2 < nT + \Delta \\ 0 & \text{elsewhere} \end{cases} \]

\[ S_x(\omega) = \int_{-\infty}^{\infty} R_x(2) e^{-j\omega t} dt = H(\omega) \]

\[ = 4\sigma^2 \frac{\sin^2(\omega T)}{\omega^2 T} + A(\omega) \]

where

\[ A(\omega) = \int_{-\infty}^{\infty} dt e^{-j\omega t} \left\{ \sum_{n=-\infty}^{\infty} \frac{\Delta}{2\Delta} \left( 1 - \frac{|2nT+\Delta|}{\Delta} \right) \right. \]

\[ \begin{cases} \frac{\Delta}{2} & \text{if } nT - \Delta < 2 < nT + \Delta \\ 0 & \text{elsewhere} \end{cases} \]

without justification

\[ = \frac{\Delta^2}{T} \sum_{m=-\infty}^{\infty} \int_{nT-\Delta}^{nT+\Delta} (1 - \frac{|2nT+\Delta|}{\Delta}) e^{-j\omega t} dt \]

after a lot of monkeyshines

\[ = \frac{\Delta^2}{T} \sum_{m=-\infty}^{\infty} \frac{e^{-j\omega nT}}{\omega^2} 4\sin^2(\frac{\omega T}{2}) \]
\[ S_x(w) = \frac{4 \sin^2 \left( \frac{\omega \theta}{2} \right)}{\omega^2 T} \left\{ \sigma^2 + \frac{2 \pi m^2}{T} \sum_{m=-\infty}^{\infty} \delta(w - \frac{2 \pi m}{T}) \right\} \]

Following Lighthill,

\[ \sum_{m=-\infty}^{\infty} e^{-j \omega m T} = \frac{2 \pi}{T} \sum_{m=-\infty}^{\infty} \delta(w - \frac{2 \pi m}{T}) \]

So

\[ S_x(w) = \frac{4 \sin^2 \left( \frac{\omega \theta}{2} \right)}{\omega^2 T} \left\{ \sigma^2 + \frac{2 \pi m^2}{T} \sum_{m=-\infty}^{\infty} \delta(w - \frac{2 \pi m}{T}) \right\} \]
7. \( X_r(w) = \int_0^T x(t) e^{-jw t} dt \)

\[
S_x(w, T) = \frac{1}{T} |X_r(w)|^2 = \frac{1}{T} \int_0^T \int_0^T x(t)^* x(t') e^{jw (t-t')} e^{jw t} dt' dt
\]

Let \( t' = t - T \)

\[
= \frac{1}{T} \int_0^T \int_0^{T-T} x(t)^* x(t+T) e^{-jw t} dt dt
\]

[Diagram]

Changing order of integration and taking \( E[S_x(w, T)] \)

\[
E[S_x(w, T)] = \frac{1}{T} \left( \int_0^T R_x(t) e^{-jw t} dt + \int_{-T}^0 R_x(t) e^{-jw t} dt \right)
\]

\[
= \frac{1}{T} \left( \int_{-T}^0 (T+t)R_x(t) e^{-jw t} dt + \int_0^T (T-t)R_x(t) e^{-jw t} dt \right)
\]

\[
= \int_{-T}^T \left( 1 - \frac{1T}{T} \right) R_x(t) e^{-jw t} dt
\]

Now, taking the limit as \( T \to \infty \) we have

\[
\lim_{T \to \infty} E[S_x(w, T)] = \int_{-\infty}^{\infty} R_x(t) e^{-jw t} dt = S_x(w)
\]

if it exists
\[ \frac{V(j\omega)}{V_i(j\omega)} = \frac{1}{1+j\omega} = \frac{1}{1+j\omega} = H(j\omega) \]

\[ S_{V_i}(\omega) = E^2 \]

\[ S_V(\omega) = S_{V_i}(\omega) |H(j\omega)|^2 = \frac{E^2}{1+\omega^2} \]

\[ R_V(\tau) = \frac{E^2}{2} e^{-12\tau} \]

Now \( e(t) = a \sin(t + \phi) + b \cos(t + \phi) \)

where \( \phi(\omega) \)

\[ R_e(\tau) = E \left[ \left\{ a \sin(t + \tau + \phi) + b \cos(t + \tau + \phi) \right\} \left\{ a \sin(t + \phi) + b \cos(t + \phi) \right\} \right] \]

\[ = E \left[ a^2 \left\{ \sin^2(t+\phi) \cos(t + \tau + \phi) + \cos(t + \phi) \sin(t + \phi) \sin(t + \tau + \phi) \right\} \right. \\
\left. + a b \left\{ \sin(t+\phi) \cos(t + \phi) \cos(t + \phi) + \cos^2(t+\phi) \sin(t + \phi) \right\} \right. \\
\left. - a b \left\{ \cos(t+\phi) \cos(t + \phi) \sin(t + \phi) - \sin^2(t+\phi) \sin(t + \phi) \right\} \right. \\
\left. + b^2 \left\{ \cos^2(t+\phi) \cos(t + \phi) - \sin^2(t+\phi) \sin(t + \phi) \right\} \right] \]
\[ \frac{a^2}{2} \cos \tau + \frac{a^2}{2} \sin \tau - \frac{a^2}{2} \sin \tau - \frac{b^2}{2} \cos \tau \]

\[ = \frac{a^2 + b^2}{2} \cos \tau \]

Now
\[ e_0(t) = v(t) + e(t) \]
\[ R_e(t) = Re(t) + Re(t) + Re(t) + Re(t) \]

since \( e(t) \) and \( v(t) \) are independent.

\[ Re(t) = m_e m_v = 0 \]
\[ Re(t) = m_v m_e = 0 \]

since \( m_e = m_e^* = 0 \)

Now
\[ R_0(t) = Re(t) + Re(t) \]
\[ = \frac{a^2 + b^2}{2} \cos \tau + \frac{a^2}{2} \cos \tau \]

And
\[ S_e(\omega) = \frac{(a^2 + b^2)}{2} \left[ \delta(\omega + 1) + S(\omega - 1) \right] + \frac{a^2}{\omega + \omega^2} \]
\[ \frac{a^2 + b^2 + c^2}{2} \]

\[ \text{Re} \theta(I) \]

\[ S(\omega) \]

\[ \text{Impulse} \]

\[ \text{strength} = \pi \left( \frac{a^2 + b^2}{2} \right) \]
7-10 We are given

\[ f_E(e) = 0 \quad \forall |e| > e_m, \quad e_m < \Delta < \frac{T}{2} \]

I assume that

\[ \Delta + 2e_m < \frac{T}{2} \]

Then the center lobe of the autocorrelation function is identical to the center lobe of the no jitter case and is given by

\[ f(t) = \frac{1}{T} \left( \delta(t) + \delta(t - 2\pi T) \right) \]

Since the \( \{ e_i \} \)'s are identically distributed, we may conclude:

1) the side lobes are centered on \( \tau = nT \)
2) " " " " have identical shapes

In light of this we proceed to calculate the autocorrelation function for one of the side lobes using ensemble averaging.

Let \( \tau' = \tau - nT \)

\[ R_x(\tau') = 0 \quad \Delta + 2e_m \leq |\tau'| < T - (\Delta + 2e_m) \]

We need not worry about the upper limit on
Here because that corresponds to the next line over.

A particular ensemble member is shown below:

\[ a - t' \rightarrow a_i \]
\[ t_1 \quad t_2 \quad t_2 + \epsilon_j \]

\[ t_1 + \epsilon_i \quad t_2 + \epsilon_i + \epsilon_j \]

\[ t \neq j \]

\[ l = \Delta - \left| t_2 - mT + \epsilon_j - (t_i + \epsilon_i) \right| = \Delta - \left| t' + \epsilon_j - \epsilon_i \right| \]

Since the \( \{a_i\} \) are independent \( \forall \ i \neq j \)

\[ R_i(t') = \frac{m^2}{T} E[f \text{ at } t'] \]
\[ = \frac{m^2}{T} \{ \Delta - E \left| t' + \epsilon_j - \epsilon_i \right| \} \text{ Ans.} \]

Let \( X = |t' + \epsilon_j - \epsilon_i| \)

Then to find \( E[X] \), one finds \( F_X(x) \) by using the product of two \( f_E \)'s, one appropriately shifted by \( t' \) (product of independent) and integrating \( f_E \) over the appropriate area. Then

\[ f_X(x) = \frac{2}{L} F_X(x) \]

\[ E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx \]
\( b) \ X(t) = \sum_{m=-\infty}^{\infty} a_m \left[ u(t + nT + \epsilon_m) - u(t + nT + \epsilon_m + T) \right] \)

since the process is ergodic we may specify (for calculation purposes) that \( L \) is the maximum value.

\[ X_L(\omega) = \sum_{\omega} X(\theta) e^{-j\omega \theta} d\theta \]

\[ = \sum_{\omega} \left[ e^{-j\omega (\theta_0 + nT + \epsilon_m)} - e^{-j\omega (\theta_0 + nT + \epsilon_m + T)} \right] e^{-j\omega \theta} d\theta \]

\[ = \sum_{m=0}^{N} a_m \int_{\theta_0 + mT + \epsilon_m}^{\theta_0 + (m+1)T + \epsilon_m} e^{-j\omega \theta} d\theta, \quad L = \theta_0 + NT + \epsilon_m \]

\[ = \sum_{m=0}^{N} \frac{a_m}{j\omega} \left[ e^{-j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})} - e^{-j\omega (\theta_0 + nT + \epsilon_m)} \right] \]

\[ = \sum_{m=0}^{N} \frac{a_m}{j\omega} e^{-j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})} \frac{2 \sin \frac{\omega T}{2}}{\omega} \]

\[ |X_L(\omega)|^2 = \frac{4 \sin^2 \frac{\omega T}{2}}{\omega^2} \sum_{m=0}^{N} a_m e^{-j\omega (\theta_0 + nT + \epsilon_m)} e^{j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})} \]

In the following we take advantage of the independence of \( \epsilon_m \) and \( \epsilon_n \) with respect to each other and themselves.

\[ E[X_L(\omega)] = \frac{E \sum_{m=0}^{N} a_m e^{-j\omega (\theta_0 + nT + \epsilon_m)} e^{j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})}}{L} \]

\[ = \frac{4 \sin^2 \frac{\omega T}{2}}{\omega^2} \sum_{m=0}^{N} E[a_m \epsilon_m] e^{-j\omega (\theta_0 + nT + \epsilon_m)} e^{j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})} \frac{E[e^{j\omega (\theta_0 + nT + \epsilon_m + \frac{T}{2})}]}{E[e^{-j\omega (\theta_0 + nT + \epsilon_m)}]} \]
\[ E[x_n x_K] = \begin{cases} \sigma^2 + m^2 & n = k \\ m^2 & n \neq k \end{cases} \]

\[ E[e^{-j\omega(x_n - x_K)}] = \begin{cases} 1 & n = k \\ E[e^{-j\omega x_n}]E[e^{j\omega x_K}] & n \neq k \end{cases} \]

\[ = \begin{cases} 1 & n = k \\ \Phi_E(-\omega) \Phi_E(\omega) & n \neq k \end{cases} \]

Now

\[ E[S_x(\omega, L)] = \frac{1}{2 \pi \sigma^2} \int \left( \sigma^2 + m^2 \right)(N+1) \]

\[ + m^2 \Phi_E(\omega) \Phi_E(\omega) \sum_{m=0}^{N} e^{-j\omega T(N-m)} \]

Let

\[ \sum_{m=-N}^{N} e^{-j\omega T(N-m)} = \sum_{n=-N}^{N} \left[ N+1 - |m| \right] e^{-j\omega m T} \]

and in the limit as \( N \to \infty \)

\[ = \sum_{m=-\infty}^{\infty} N e^{-j\omega m T} \]

\[ = \left[ \sum_{m=-\infty}^{\infty} N e^{-j\omega m T} \right] - N \]
Finally \((\text{as } L \to \infty)\) \(N_T \to 1\)

\[
S_x(w) = \lim_{L \to \infty} E\left[ S_x(w, L) \right] = \frac{4 \sin^2\left(\frac{L \omega}{2}\right)}{T \omega^2} \left\{ (\omega^2 + n^2) + m^2 \right\} \left[ \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{-j \omega n T} - 1 \right].
\]

Using Poisson Summation Formula:

\[
\sum_{n=-\infty}^{\infty} f(nT) e^{-j \omega n T} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} F(\omega + 2\pi n T)
\]

we set \(f(nT) = 1\) \(\forall n\) then

\[
\sum_{n=-\infty}^{\infty} e^{-j \omega n T} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi n T)
\]

Then:

\[
S_x(w) = \frac{4 \sin^2\left(\frac{L \omega}{2}\right)}{T \omega^2} \left\{ (\omega^2 + n^2) + m^2 \right\} \left[ \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \pi \delta(\omega + 2\pi n T) - 1 \right].
\]

\[
\Phi_E(w) = E\left[ e^{j \omega X} \right] = \int_{-\pi}^{\pi} e^{j \omega x} f_E(x) \, dx = \frac{2}{\pi \omega} \sin \frac{\omega \pi}{2}
\]

\(\Phi_E(w) \Phi_E(-w) = \left| \Phi_E(w) \right|^2\)
\[ \Phi_e(-\omega) = -\frac{2}{\lambda \omega} \sin\left(\frac{\omega \Delta}{2}\right) = \frac{2}{\lambda \omega} \sin\frac{\omega \Delta}{2} = \Phi_E(\omega) \]

\[ S_x(\omega) = \frac{4 \sin^2\left(\frac{\omega \Delta}{2}\right)}{\pi \omega^2} \left[ \delta^2 + m^2 \left[ -\frac{4}{\pi^2 \omega^2} \sin^2\left(\frac{\omega \Delta}{2}\right) \right. \right. \]

\[ \left. + \frac{4}{\pi^2 \omega^2} \sin^2\left(\frac{\omega \Delta}{2}\right) \right] \frac{2 \pi}{\Gamma} \sum_{m=-\infty}^{\infty} \delta(\omega + \frac{2\pi m}{T}) \]

For purposes of plotting, if \( \omega \Delta \ll \Delta \) and treat the \( \sin^2\left(\frac{\omega \Delta}{2}\right) \)

\[ \sin^2\left(\frac{\omega \Delta}{2}\right) \approx \left(\frac{\omega \Delta}{2}\right)^2 \]

for the center portion of \( S_x(\omega) \)

\[ S_x(\omega) \approx \frac{4 \sin^2\left(\frac{\omega \Delta}{2}\right)}{\pi \omega^2} \left(0 + \frac{2 \pi}{\Gamma} \sum_{m=-\infty}^{\infty} \delta(\omega + \frac{2\pi m}{T}) \right) \]

---

The impulses' strengths are:

- \( 2\pi \left(\frac{\omega \Delta}{2}\right)^2 \)
- \( \frac{2m^2}{\pi} \sin^2\left(\frac{\omega \Delta}{2}\right) \)
d) Now we consider the side lobes and the sketch below.

The reason for the regions between A and B, and between B and C remaining unchanged is that the integration to be justified.

\[ E \left| T' + E(e_j - e_i) \right| = \left| T' + E(e_j - e_i) \right| = \left| T' \right| \]

Note: In part (b) the red lines become straight lines connecting their end points as shown above.
1) For the side length $c$, we have:

$$c^2 = a^2 + b^2$$

To simplify, the argument for $a$ is:

$$a = \sqrt{c^2 - b^2}$$

The diagram shows the right angle at point $C$. The sides are labeled $a$, $b$, and $c$.

2) The scenario involving $a$ and $b$ is similar to a right triangle where $c$ is the hypotenuse. The problem states:

With given values, $a$ and $b$ were used for $c$, $a$ while $b$ pertains to $b$.

For further details, the text mentions an increase in the squared side, leading to:

$$c^2 = a^2 + b^2$$

with an increase for $c$. However, the exact values are not provided.
\[ E = \sum (c^2 - 1) + c_1 \cdot \Delta + \frac{c_2}{T} \]

\[ \text{Expression for } E \text{ under certain conditions.} \]

In subsequent lines, there is a paragraph of text that is not legible due to handwriting.

The above discussion for the attachment of elements.

The notations are defined as follows:

\[ \text{Definition of } E \text{ with } E = \sum c_i \text{ for } i = 1, 2, \ldots, k \text{ and } \Delta > y. \]

Handwritten notes and diagrams are present in the image.