Signal Detection in White Noise:

Previously we have discussed the hypothesis testing problem and the estimation problem. We have also discussed the representation of random processes, in particular, the Karhunen-Loeve expansion. We now turn our attention to the application of these results to the detection and estimation problem for waveforms.

In the first lecture we discussed the various levels of detection and estimation problem. They are reviewed in section 4.1 of VanTrees. The first problem we shall consider is the detection of a known signal in white noise.

**Binary Signal Detection in White Noise**

For this case we first consider the detection of the presence or absence of a known signal in white Gaussian noise.

We assume that the signal $s(t)$ has unit energy. Thus

$$E_s = \int_0^T s^2(t) \, dt = 1 = \|s\|^2.$$  

The noise is additive with PSD

$$s_n(\omega) = \frac{N_0}{2}, \quad \text{all } \omega.$$
Thus we have an hypothesis testing problem:

\[ \begin{align*}
H_1 : \ r(t) &= \text{ES}(t) + \text{W}(t) ; \ 0 \leq t \leq T \\
H_0 : \ r(t) &= \text{W}(t) ; \ 0 \leq t \leq T
\end{align*} \]

where \( \text{ES} \) is the energy of the received signal in the absence of noise.

We shall use two concepts from our previous discussions. The first is the expansion of random processes in terms of a cone set, \( \mathcal{F} \), \( \text{u} \). The second is the determination of a sufficient statistic. Using these tools, the problem reduces to one studied previously.

We first choose a suitable cone set for the representation of \( \text{W}(t) \). We choose

\[ \Phi_i(t) = s_i(t) \]

and the remaining function are chosen to be orthogonal to \( s_i(t) \). Thus

\[ \left\{ \begin{align*}
\int_0^T s_i(t) \phi_j(t) \, dt &= 0, & j = 2, 3, \ldots \\
&= 1, & j = 1
\end{align*} \]
Now consider the expansion coefficients for \( r(t) \) under the two hypotheses:

**H₀:** \( r(t) = w(t) \)

\[
\mathbf{r}_i = \int_0^T r(t) \phi_i(t) \, dt = \int_0^T w(t) \phi_i(t) \, dt = \mathbf{w}_i
\]

The set \( \mathbf{w}_i \) are statistically independent Gaussian random variables.

**H₁:** \( r(t) = \mathbf{v}_E s(t) + w(t) \)

\[
\mathbf{r}_i = \int_0^T \left[ \mathbf{v}_E s(t) + w(t) \right] \phi_i(t) \, dt
\]

\[
= \mathbf{v}_E \int_0^T s(t) \phi_i(t) \, dt + \int_0^T w(t) \phi_i(t) \, dt
\]

\[
\begin{cases}
\mathbf{v}_E \quad & i = 1 \\
0 \\ & i \neq 1
\end{cases}
\]

*Thus, the only coefficient that depends upon which hypothesis is true is \( r_1 \). Hence*

\[
\begin{cases}
r_1 = \{ \mathbf{v}_E + \mathbf{w}_1 \}, & H₁ \\
\mathbf{w}_1, & H₀
\end{cases}
\]

\[
\begin{cases}
r_i = \mathbf{w}_i, & H₁ \quad H₀ \\
\end{cases} \\
i = 2, 3, \ldots
\]
Since $r_1$ is the only coefficient which is hypothesis dependent, we have our sufficient statistic; we observe that

$$e = r_1, \quad y = (r_2, r_3, \ldots)$$

The receiver

Since we only need $r_1$ to make the decision we can implement the receiver in one of the two ways, (see p. 75).

(a)

(b) $h(t)$

These receivers are known as correlation receivers.

* The correlation receiver in the form of b) is known as a matched filter for white Gaussian noise, (see notes of previous course and prob 3.3.1).
The simple binary detection problem in white noise has been reduced to a simple hypothesis testing problem which has been studied previously.

\[
\begin{align*}
H_0 & : r_i = w_i = \epsilon \\
H_1 & : r_i = \sqrt{\epsilon} + w_i = \epsilon
\end{align*}
\]

The performance of the test was measured by the ROC curves with parameter \( d \), where

\[
d^2 = \frac{\left[ E(e/H_0) - E(e/H_1) \right]^2}{\text{Var}(e/H_0)} \quad (\text{see eq. 2.334})
\]

Hence:

\[
E[e/H_0] = E[h/H_0] = 0
\]

\[
E[e/H_1] = E[h/H_1] = \sqrt{\epsilon}
\]

\[
\text{Var}(e/H_0) = \frac{N_0}{2}
\]

Thus

\[
d^2 = \frac{2\epsilon}{N_0}
\]

So we have the ROC curves on pp. 38, 250. These are the same curves with \( d = \sqrt{2\epsilon/N_0} \).
Alternate derivation

We now derive the same results without recognizing the sufficient statistic. We shall take any \( \mathcal{C} \) at \( \Phi(t) \) to start the problem. Again

\[
\begin{align*}
H_0 & : r(t) = w(t) \\
H_1 & : r(t) = \sum_{i=1}^{k} \phi_i(t) + w(t)
\end{align*}
\]

we expand \( r(t) + \sqrt{E} s(t) \) and use the first \( k \) terms of the expansions:

\[
\begin{align*}
\hat{r}_k(t) & = \sum_{i=1}^{k} \hat{r}_i \phi_i(t) \\
\hat{s}_k(t) & = \sum_{i=1}^{k} \hat{s}_i \phi_i(t)
\end{align*}
\]

where \( \hat{r}_i = (r, \phi_i), \hat{s}_i = (\sqrt{E}s, \phi_i) \)

Then

\[
\lim_{k \to \infty} \| r - \hat{r}_k \| = 0 \quad \lim_{k \to \infty} \| \sqrt{E}s - \hat{s}_k \| = 0
\]

Now we look at the coefficients of \( r(t) \):

\[
\begin{align*}
H_0: & \quad \hat{r}_i = (r, \phi_i) = (w, \phi_i) = \hat{w}_i \\
H_1: & \quad \hat{r}_i = (r, \phi_i) = (\sqrt{E}s + w, \phi_i) = (\sqrt{E}s, \phi_i) + (w, \phi_i) = \hat{s}_i + \hat{w}_i
\end{align*}
\]

Since the noise is Gaussian and white, \( \hat{w}_i \) are independent, zero mean random variables.
Thus

\[ E[\hat{r}_i \mid H_0] = 0, \quad E[\hat{r}_i \mid H_1] = s_i^* \]

\[ \text{Var} [\hat{r}_i \mid H_{0,1}] = \frac{N_0}{\pi} \]

Now we form the LHR

\[ \Lambda(k) = \frac{P_{R_k \mid H_1} (R_k \mid H_1)}{P_{R_k \mid H_0} (R_k \mid H_0)}\]

\[ = \frac{\prod_{i=1}^{k} \frac{1}{\pi N_0} e^{-[R_i - S_i]^2/N_0}}{\prod_{i=1}^{k} \frac{1}{\pi N_0} e^{-R_i^2/N_0}} \]

Taking the log and cancelling common terms yields

\[ \ln \Lambda(k) = \frac{2}{N_0} \sum_{i=1}^{k} R_i S_i - \frac{1}{N_0} \sum_{i=1}^{k} S_i^2 \]

Now using Parseval's formula

\[ \sum_{i=1}^{k} R_i S_i = \int_{0}^{T} r_k(t) s_k(t) dt \quad \sum_{i=1}^{k} S_i^2 = \int_{0}^{T} s_k^2(t) dt \]

\[ \left( \rightarrow \int_{0}^{T} r \ast s dt \right) \quad \left( \rightarrow E \right) \]

and extracting the limit \( k \to \infty \)

\[ \lim_{k \to \infty} \Lambda(k) = \Delta = \frac{2}{N_0} \int_{0}^{T} r(t) (\sqrt{2} \ast s(t)) dt - \frac{E}{N_0} \]

\[ \Delta = \frac{2 \sqrt{2}}{N_0} \int_{0}^{T} r(t) s(t) dt - \frac{E}{N_0} \]
Thus introducing the threshold \( \gamma \) we have

\[
\int_{0}^{T} r(t) s(t) \, dt = \frac{N_0}{2 \sqrt{2}} \ln N + \frac{\gamma}{2 N_0} \frac{1}{\sqrt{2}}
\]

as the test for the simple binary detection problem. Again we see that we simply compare

\[ L = \int_{0}^{T} r(t) s(t) \, dt \]

to a threshold.

**General Binary Detection Problem in WGN:**

We now consider the problem of deciding which of two known signals is present in WGN.

\[ H_1 : \quad r(t) = \sqrt{E_1} s_1(t) + w(t) \quad 0 \leq t \leq T \]
\[ H_0 : \quad r(t) = \sqrt{E_0} s_0(t) + w(t) \]

We will assume that the signals but not necessarily orthogonal. Thus

\[
||s_1|| = ||s_0|| = 1, \quad (s_1, s_0) = \rho
\]

\[
|\rho| = \left| \int s_0^T s_1(t) s_1(t) \, dt \right| \leq \left[ \int s_0^T s_1^2(t) \, dt \int s_1^T s_1^2(t) \, dt \right]^{1/2} = 1
\]

\[ \therefore \rho^2 \leq 1 \]
The first two \textit{con} functions are chosen to be
\[
\phi_1(t) = s_i(t), \quad 0 \leq t \leq T
\]
\[
\phi_2(t) = \frac{s_0(t) - (s_0 s_i) s_i(t)}{\|s_0 - (s_0 s_i) s_i\|} = \frac{s_0(t) - P s_1(t)}{\sqrt{1 - P^2}}, \quad 0 \leq t \leq T
\]

*recall the Gram-Schmidt procedure*

Again the remaining members of the \textit{con} set are arbitrary except $(\phi_i, \phi_j) = \delta_{ij}$. We again expand $v(t)$ under both hypotheses:

The only two coefficients which are hypothesis dependant are $r_1$ and $r_2$:

\[
r_i = \int_0^T s_i(t) \phi_i(t) \, dt, \quad i = 1, 2, \ldots
\]

\underline{H_1}:

\[
r_1 = (r, \phi_1) = (\bar{v} e, s_1 + w, \phi_1) = \bar{v} e (s_1 \phi_1) + w
\]

\[
= s_{1i} + w_i
\]

\[
r_2 = (r, \phi_2) = (\bar{v} e, s_1 + w, \phi_2) = s_{1i} + w_2
\]

\underline{H_0}:

\[
r_1 = (r, \phi_1) = s_{0i} + w_i
\]

\[
r_2 = (r, \phi_2) = s_{02} + w_2
\]

And

\[
r_2 = w_i, \quad i = 3, 4, \ldots \quad H_0, H_1
Thus

\[ E \left[ r_i / H_j \right] = S_{ij}, \quad i = 1, 2, \ldots, j = 0,1 \]

\[ E \left[ r_i / H_j \right] = 0, \quad i = 3, 4, \ldots, j = 0,1 \]

and

\[ \text{Var} \left[ r_i / H_j \right] = \frac{N_0}{2} \quad \text{all } i, j \]

We can express these results in matrix notation and use the results of section 2.6.

\[ S_{0} \,(= \, M_{0}) \triangleq E[ z / H_{0} ] = \begin{bmatrix} S_{01} \\ S_{02} \end{bmatrix} \]

\[ S_{1} \,(= \, M_{1}) \triangleq E[ z / H_{1} ] = \begin{bmatrix} S_{11} \\ S_{12} \end{bmatrix} \]

\[ k_1 = k_2 = k = \begin{bmatrix} N_{0}/2 & 0 \\ 0 & N_{0}/2 \end{bmatrix} = \frac{N_0}{2} I \]

Thus

\[ k^{-1} = \Omega = \frac{2}{N_0} I \]

From (2.827) and (2.828) we can state LRT for the decision

\[ \Delta m^{*} \in R_{+} \quad \frac{\lambda}{N_0} + \frac{1}{N_0} \left( M_{1} + \Omega M_{1} - N_{0} \Omega M_{0} \right) \]

which is

\[ \Delta m = S_{1} - S_{0} \]

\[ \frac{2}{N_0} \left[ S_{1} - S_{0} \right]^{T} R_{+} \frac{1}{N_0} \Delta \lambda + \frac{1}{N_0} \left( 15_{1}^{2} - 15_{0}^{2} \right) \]
which finally becomes

\[ R^* (s_1 - s_0) = \frac{N_0}{2} \ln 2 + \frac{1}{2} (15_1^2 - 15_0^2) \]

Thus is Z space, \( Z = \frac{1}{2} (R_1 R_2) \), we have a line defined by

\[ R^* (s_1 - s_0) = \frac{N_0}{2} \ln 2 + \frac{1}{2} (15_1^2 - 15_0^2) \]

A very important observation should be made here. In order to make a decision only the difference between \( s_0 \) and \( s_1 \) is required. So consider

\[ S_1 - S_0 = \begin{bmatrix} s_{11} - s_{01} \\ s_{12} - s_{02} \end{bmatrix} = \int_0^T (V_{E_1} s_{11} + V_{E_0} s_{01} (t)) \phi_1 (t) dt \]

\[ S_1 - S_0 = \begin{bmatrix} s_{11} - s_{01} \\ s_{12} - s_{02} \end{bmatrix} = \int_0^T (V_{E_1} s_{11} + V_{E_0} s_{01} (t)) \phi_2 (t) dt \]

Let \( S_o (t) = V_{E_1} T_1 (t) + V_{E_0} s_{01} (t) \). We shall return to this shortly.

In Z space the points \( S_1 \) and \( S_0 \) can be plotted and the decision is based upon which side of the line the data point is on.
It can be shown easily that any line in $\mathbb{R}^2$ of the form

$$R^+(s_i - s_0) = k$$

is always perpendicular to the vector $(s_i - s_0)$, (show this at home).

Now let us return to the observation made on the previous page. Let us now consider the difference function, $s_\Delta(t)$. We first normalize

$$s_\Delta(t) = \frac{s_\Delta(t)}{\|s_\Delta(t)\|} = \frac{\sqrt{E_1 s_1(t) - V E_0 s_0(t)}}{(E_1 - 2P \sqrt{E_0 E_0 + E_0})^{1/2}}$$

and now choose $\phi_i(t) = s_\Delta(t)$ and all other $\phi_i$ to be orthogonal to $\phi_i(t)$. This corresponds to a linear transformation of the $\mathbb{R}^2$ plane.

$$\begin{bmatrix} E_1 \\ s_0 \end{bmatrix} = A \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$
* Note that knowledge of \( y \) doesn't help make the decision and only \( E \) is needed. Once again we have deduced what the sufficient statistic is. Since we have \( E \) we can now investigate the performance of the test. It is the same as before but \( d \) is now different. To calculate \( d \) we need

\[
E \left[ e/\mathcal{H}_0 \right] = E \left[ (r_1, \phi_1)/\mathcal{H}_0 \right] = E \left[ (r, s_0)/\mathcal{H}_0 \right] \\
= \sqrt{\mathcal{E}_0} (s_0, \bar{s}_0) = \frac{\sqrt{\mathcal{E}_0} (s_0, \bar{s}_0) - \sqrt{\mathcal{E}_0} s_0}{(E_1 - 2P \sqrt{\mathcal{E}_0} \bar{e}_1 + E_0)^{1/2}} < D \\
= \frac{\sqrt{\mathcal{E}_0} \bar{e}_1 - E_0}{D}
\]

Likewise

\[
E \left[ e/\mathcal{H}_1 \right] = \frac{E_1 - P \sqrt{\mathcal{E}_0} \bar{e}_1}{D}
\]

and \( \text{Var} \left[ e/\mathcal{H}_1 \right] = \frac{\mathcal{H}_0}{\alpha} \) (show at home)
Thus

\[ d^2 = \frac{2}{N_0} (E_i + E_o - 2P \sqrt{E_i E_o}) \]

The ROC curve are as before and the optimum receiver is

\[ r(t) \xrightarrow{h(t)} S(t) \xrightarrow{t=T} \text{H} \]

Sample at \( t=T \)

**Best Signals**

It should be clear that if we have our choice of signals to choose we would want to make \( b \) as large as possible. Since \( E_i \) and \( E_o \) is fixed we make

\[ P = -1 \Rightarrow d^2 = \frac{2}{N_0} (E_i + E_o + 2 \sqrt{E_i E_o}) \]

but \( P = (S_i, S_o) = -1 \) \( \iff \)

\[ S_i(t) = -S_o(t) \]

\[ \text{eg.} \]

\[ \begin{array}{c}
S_i(t) \\
T
\end{array} \xrightarrow{t=\frac{T}{2}} \]

\[ \begin{array}{c}
S_o(t) \\
\frac{T}{2}
\end{array} \]

* For this case (know signals) we do not get the answer \( (S_i, S_o) = 0 \)
**Decision Criteria:**

a) If the criterion is **minimum error probability** ($C_{ef} = 1 - \frac{1}{2}f^2$) and $P_0 = P_1$. Then $y = 1$ the decision line is the perpendicular bisector of $S_1 - S_0$ (recall $\frac{1}{2}$). Under these conditions we have a **minimum distance receiver** i.e. choose the smallest of $\left\{ \frac{S_1 - S_0}{E_0} \right\}$.

\[ P_r(e) = \text{erfc}_\ast \left( \frac{1}{2} \right) \]

b) If in addition to above, $E_0 = E_1$, then the decision line goes through $E = 0$ and we can simply choose the signal which is most correlated to $n(t)$.

For this case choose the largest of $\left\{ (r, S_1), (r, S_0) \right\}$.
M-ary Detection in WGN:

We now assume that there are $M$ hypotheses and we have to choose one of them. Thus

$$H_i: \quad r(t) = \sqrt{\gamma} s_i(t) + w(t), \quad 0 \leq t \leq T \quad i = 0, \ldots, M-1$$

we assume that there exists a correlation matrix (Gramian matrix)

$$P = \left[ p_{ij} = \int_{0}^{T} s_i(t) s_j^*(t) dt \right]$$

such that $\|s_i\|^2 = p_{ii} = 1$, $(s_i, s_j) = p_{ij}.$

* This problem reduces to the $M$-hypothesis problem in chap. 2. (pp. 46-52)

The first step in the procedure is to find a suitable basis for the representation of $r(t)$. To accomplish this task we use the Gram-Schmidt procedure on the set $\{s_i(t)\}$:

$$\begin{cases} 
\phi_1(t) = c_{11} s_1(t) \\
\phi_2(t) = c_{12} s_1(t) + c_{22} s_2(t) \\
\vdots \\
\phi_n(t) = c_{1n} s_1(t) + \cdots + c_{nn} s_n(t)
\end{cases} \quad (c_{11} = 1)$$
One of two things will occur.

a) If $|p_1| > 0$, then all signals are CI and we will obtain $N = M$ orthonormal signals $\phi_i(t)$. 

b) If $|p_1| = 0$, then $N < M$ CI signals will be obtained. The remaining $M - N$ signals will be LD upon the first $N$.

* The decision space will be at most $M$ but can be reduced if the signals are LD.

Now we expand $r(t)$ using

$$r_i = \int_0^T r(t) \phi_i(t) \, dt, \quad i = 1, \ldots, N$$

and

$$E[r_i / A_j] = \delta_{ij}, \quad i = 1, \ldots, N$$

The remainder of the solution is obvious from discussions in chap. 2. We simply define $M - 1$ LR's and have $M - 1$ inequalities to be satisfied. We now turn to some special cases.