1) **Lower bounded random walk.** Consider a game in which players bet $1 to win $1 with probability $p$ and lose their bets with probability $q = 1 - p$. The wealth of a player as a function of time is a random process. If the player’s wealth at time $t$ is $w(t)$ (which denotes a realization of the random variable $W(t)$), the wealth at time $t+1$ is either $w(t) + 1$ or $w(t) - 1$. Moreover, the probability of the wealth increasing to $w(t) + 1$ is $p$ and the probability of the wealth decreasing to $w(t) - 1$ is $q$. We write this as

\[
\begin{align*}
P[W(t+1) = w(t) + 1 | W(t) = w(t)] &= p, \\
P[W(t+1) = w(t) - 1 | W(t) = w(t)] &= q. 
\end{align*}
\]

The first equation, e.g., is read as “the probability of $W(t+1)$ taking the value $w(t) + 1$, given $W(t) = w(t)$ is $p$.” The expression in (1) is true as long as $W(t) \neq 0$. When $W(t) = 0$ the gambler is ruined and $W(t+1) = 0$.

A rather sophisticated, yet sometimes useful way of expressing this fact is

\[P[W(t+1) = 0 | W(t) = 0] = 1. \tag{2}\]

We saw in class that if $p > 1/2$ then it is likely that the sample paths $w(t)$ of the random process diverge making this a rather good game to play. In this exercise $p$ can take any value. This process can be called a lower bounded random walk. Wealth can be reinterpreted as position on a line and wealth variations as steps taken randomly to left and right. The origin is home, in that if the walker reaches 0 it stays there. It is asked that:

A) **Simulation of a process realization.** Write a function that accepts as parameters the probability $p$, the initial wealth $W(0) = w_0$ and a maximum number of bets $T$. The function returns a vector of length at most $T + 1$ containing the wealth’s history $w(0), \ldots, w(T)$ randomly computed according to the probabilities in (1) and (2). If the wealth is depleted at time $t < T$, that is, if $w(t) = 0$ for some $t < T$, the function returns a vector of length $t + 1$ with the wealth’s history up to time $t$, i.e., $w(0), \ldots, w(t)$. Optionally, you can also return a boolean variable to distinguish between a run that resulted in a broken player and one that did not. This might be useful for parts B-E. Show plots with simulated processes for $w_0 = 20$, $T = 10^3$ and $p = 0.25$, $p = 0.5$ and $p = 0.75$.

B) **Probability of reaching home.** Fixing $p = 0.55$ and $w_0 = 10$ compute the probability $B(p, w_0)$ of eventually reaching home (going broke in the betting context), that is the probability of having $W(t) = 0$ for some $t$. Notice that because once $W(t) = 0$ wealth stays at 0 this probability can be written as the limit

\[B(p, w_0) = \lim_{t \to \infty} P[W(t) = 0 | W(0) = w_0]. \tag{3}\]

Strictly speaking, you would need to run the simulation forever to make sure the gambler does not run out of money. However, you can truncate simulations at time $T = 100$ for this exercise. With this approximation you would be aiming to compute the probability of reaching home between times 0 and $T$, which we assume approximates the probability of reaching home between times 0 and $\infty$ reasonably well. Put differently, we are assuming that $P[W(T) = 0 | W(0) = w_0]$ for $T = 100$ is a good approximation of the limit in (3). To estimate $P[W(T) = 0 | W(0) = w_0]$ we run the simulation code of part A multiple times. Each of these runs results in a wealth path $w_n(t)$, we then define the indicator function $I\{w_n(T) = 0\}$ which equals 1 if wealth at time $T$ is $w_n(T) = 0$ and 0 if not. The probability of reaching home is then estimated as ($N$ is the number of simulations ran)

\[\hat{B}_N(p, w_0) = \frac{1}{N} \sum_{n=1}^{N} I\{w_n(T) = 0\}. \tag{4}\]

The expression in (4) is just the average number of times home was reached across all experiments. The function $I\{w_n(T) = 0\}$ is called the indicator function of the event $w_n(T) = 0$ because it “indicates” the event by taking the value 1.

To compute $\hat{B}_N(p, w_0)$ you need to decide on a number of experiments $N$. The more experiments $N$ you run the more accurate your estimation. Alas, the larger you need to wait. Report your probability estimate and the number of experiments $N$ used. Explain your criteria for selecting $N$.
C) Probability of reaching home as a function of initial wealth. We want to study the probability of reaching home as a function of initial wealth. Fix \( p = 0.55 \) and vary initial wealth between \( w_0 = 1 \) and \( w_0 = 20 \). Show a plot of your probability estimates \( \hat{B}_N(p, w_0) \) as a function of initial wealth. The number of experiments \( N \) run to compute probability estimates for different initial wealths need not be the same.

D) Probability of reaching home as a function of \( p \). The goal is to understand the variation of the probability of reaching home for different values of the probability \( p \). Fix \( w_0 = 10 \) and vary \( p \) between 0.3 and 0.7 – increments 0.02 should do. Show a plot of your probability estimates \( \hat{B}_N(p, w_0) \) as a function of \( p \). You should observe a fundamentally different behavior for \( p < 1/2 \) and \( p > 1/2 \). Comment on that.

E) Time to reach home. Fix \( p = 0.4 \). With this value of \( p \) it is possible to see that gamblers eventually deplete their wealth independently of their initial wealth \( w_0 \). This is something remarkable, despite the process being random it is possible to say that \( W(t) \) eventually becomes 0. This needs to be qualified, though. Unlikely as it may be there is a chance of winning all hands. Of course, the probability of this happening becomes smaller as the gambler plays more hands. What we can say about a lower bounded random walk is that with probability 1, wealth \( W(t) \) approaches 0 as \( t \) grows. More formally, the limit \( \lim_{t \to \infty} W(t) \) satisfies

\[
P \left( \lim_{t \to \infty} W(t) = 0 \right) = 1. \tag{5}\]

We say that \( \lim_{t \to \infty} W(t) = 0 \) almost surely. Different wealth paths are possible, but almost all of them result in a broken gambler. If we think of probabilities as measuring the likelihood of an event, the measure of the event \( W(t) \neq 0 \) is asymptotically null. An important quantity here is the time at which \( W(t) = 0 \) for the first time which we can write as

\[
T_0 = \min_t (W(t) = 0). \tag{6}\]

Estimate the probability distribution of \( T_0 \) and its average value.