1) **White Gaussian noise.**

* A) **Independent values.** The autocorrelation function of the white Gaussian noise (WGN) process is

\[ R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2) \]

where \( \delta(t) \) is the Dirac delta (generalized) function that is infinite at \( t = 0 \), and zero everywhere else. This means that for different times \( t_1 \) and \( t_2 \), \( R_W(t_1, t_2) = 0 \). Therefore, the random variables \( W(t_1) \) and \( W(t_2) \) are uncorrelated. But since \( W(t) \) is a Gaussian process (GP) then uncorrelatedness implies independence, so that \( W(t_1) \) and \( W(t_2) \) are also independent.

* B) **Integral of WGN.** The Brownian motion process

\[ X(t) = \int_0^t W(u)du \]

is a GP, because it is a linear functional of a GP (WGN specifically). Using the linearity of the expectation operator and that \( \mu_W(t) = 0 \), the mean function \( \mu_X(t) \) of \( X(t) \) is

\[ \mu_X(t) = E \left[ \int_0^t W(u)du \right] = \int_0^t E[W(u)]du = \int_0^t \mu_W(u)du = 0. \]

Likewise, since \( R_W(t_1, t_2) = \sigma^2 \delta(t_1 - t_2) \) the autocorrelation function \( R_X(t_1, t_2) \) of \( X(t) \) is for \( t_1 < t_2 \)

\[ R_X(t_1, t_2) = E \left[ \left( \int_0^{t_1} W(u)du \right) \left( \int_0^{t_2} W(v)dv \right) \right] \]

\[ = E \left[ \int_0^{t_1} \int_0^{t_2} W(u)W(v)dvdu \right] \]

\[ = \int_0^{t_1} \int_0^{t_2} E[W(u)W(v)]dvdu \]

\[ = \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(u-v)dvdu \]

\[ = \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(u-v)dvdu + \int_0^{t_1} \int_{t_1}^{t_2} \sigma^2 \delta(u-v)dvdu \]

\[ = \int_0^{t_1} \int_0^{t_2} \sigma^2 \delta(u-v)dvdu = \int_0^{t_1} \sigma^2 du = \sigma^2 t_1. \]

Arguing in the exactly same way for \( t_2 < t_1 \) yields \( R_X(t_1, t_2) = \sigma^2 t_2 \), so that all in all

\[ R_X(t_1, t_2) = \sigma^2 \min(t_1, t_2). \]

Since the Brownian motion process is a GP, then for each \( t \geq 0 \), \( X(t) \) is a zero-mean Gaussian random variable with variance

\[ \text{var}[X(t)] = E[X^2(t)] - E[X(t)]^2 = R_X(t, t) - \mu_X^2(t) = \sigma^2 t. \]

So any desired probability can be obtained by suitably integrating the Gaussian pdf, for instance

\[ P[X(t) > a] = \int_a^\infty \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp \left( -\frac{x^2}{2\sigma^2 t} \right) dx. \]

* C) **Discrete time representation of WGN.** For the discrete-time process \( W_h(n) \) defined by

\[ W_h(n) = \int_{nh}^{(n+1)h} W(u)du \]
the mean function is
\[ \mu_{W_h}(n) = \mathbb{E}[W_h(n)] = \mathbb{E} \left[ \int_{nh}^{(n+1)h} W(u) \, du \right] = \int_{nh}^{(n+1)h} \mathbb{E}[W(u)] \, du = \int_{nh}^{(n+1)h} \mu_W(u) \, du = 0. \]

The autocorrelation function \( R_{W_h}(n_1, n_2) \) follows from the definition, namely
\[ R_{W_h}(n_1, n_2) = \mathbb{E} \left[ \left( \int_{n_1 h}^{(n_1+1)h} W(u) \, du \right) \left( \int_{n_2 h}^{(n_2+1)h} W(v) \, dv \right) \right] \]
\[ = \mathbb{E} \left[ \int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} W(u)W(v) \, du \, dv \right] \]
\[ = \int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} \mathbb{E}[W(u)W(v)] \, du \, dv \]
\[ = \int_{n_1 h}^{(n_1+1)h} \int_{n_2 h}^{(n_2+1)h} \sigma^2 \delta(u - v) \, du \, dv = \begin{cases} \sigma^2 h, & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases} . \]

So upon defining the discrete-time Dirac delta sequence \( \delta_d(n) \) as
\[ \delta_d(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \]
the autocorrelation function can be expressed as \( R_{W_h}(n_1, n_2) = \sigma^2 h \delta_d(n_1 - n_2) \).

**D) Simulation of Brownian motion process.** The Matlab code to perform a discrete-time simulation \( X_h(n) = \sum_{i=0}^{n} W_h(i) \) of the Brownian motion process \( X(t) \) follows. The choice of parameters is \( h = 0.01, \sigma^2 = 1, \) and \( t_{\text{max}} = 10 \). Based on our previous calculations the mean is \( \mu_{X_h}(n) = 0 \), and the variance is \( \text{var}[X_h(n)] = R_{X_h}(n,n) = \sigma^2 h \).

```matlab
clear all; close all; clc;
h=0.01; sigma=1; t_MAX=10;
W_vector=normrnd(0,sigma*sqrt(h),1,t_MAX/h);
X_vector=cumsum(W_vector);
plot(h:h:t_MAX,X_vector,'r','Linewidth',1);
xlabel('time');title(['Weiner Process Simulated, h=',num2str(h)]);
grid on; axis([0 t_MAX -5 5])
```
Weiner Process Simulated, h=0.01

Fig. 1. A simulated sample path of the Weiner process $X(t)$ using its discrete approximation. (Part D.)