ECE440 - Introduction to Random Processes

Midterm Exam

November 5, 2014

Instructions:
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 104, extra points are bonus points).
- Duration: 75 minutes.
- This exam has 9 numbered pages, check now that all pages are present.
- Show all your work, and write your final answers in the boxes when provided.

Name:______________________________

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GOOD LUCK!
1. Suppose that $X_N = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}.$$ 

To spare you of pointless calculations, if needed you may use that

$$P^3 = \begin{pmatrix} 19/32 & 13/32 \\ 39/64 & 25/64 \end{pmatrix} = \begin{pmatrix} 0.59 & 0.41 \\ 0.61 & 0.39 \end{pmatrix}.$$ 

(a) (2 points) $P[X_4 = 2 \mid X_3 = 1, X_2 = 2, X_1 = 1] = ?$

(b) (2 points) $P[X_5 = 1 \mid X_2 = 1, X_0 = 1] = ?$

(c) (2 points) $P[X_3 = 1 \mid X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2] = ?$
(d) (6 points) \( \mathbb{E} [X_7 | X_4 = 2] =? \)

(e) (8 points) Let \( N = \min \{ n > 0 : X_n = 2 \} \). \( \mathbb{E} [N | X_0 = 1] =? \)
2. (8 points) Consider a probability space \((S, \mathcal{F}, P[\cdot])\). Suppose that \(D\) and \(E\) are events in \(\mathcal{F}\) such that \(P[\{D\}] = \frac{3}{5}\) and \(P[\{E\}] = \frac{4}{5}\). From this information, is it possible to tell if \(D\) and \(E\) are mutually exclusive? Explain.

3. (8 points) Suppose that a random variable \(X\) is Poisson-distributed with parameter \(\lambda > 0\); i.e., \(P[X = x] = e^{-\lambda} \frac{\lambda^x}{x!}\) for \(x = 0, 1, 2, \ldots\). Define the random variable \(Y = qX\), where \(q\) is a number such that \(0 < q < 1\). Is \(Y\) Poisson-distributed? Justify your answer.
4. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$
P = \begin{pmatrix}
0 & a & b & 0 & 0 \\
c & 0 & d & 0 & 0 \\
e & 0 & 0 & 0 & 0 \\
0 & 0 & f & 0 & g \\
0 & 0 & 0 & h & 0
\end{pmatrix}
$$

where $a, b, c, d, e, f, g, h > 0$.

(a) (8 points) Is the Markov chain irreducible? Explain.

(b) (6 points) Is state 4 transient? Explain.
5. Suppose that \( X_N = X_1, X_2, \ldots, X_n, \ldots \) is an i.i.d. sequence of random variables, where \( P[X_1 = 1] = 1/4, P[X_1 = 2] = 1/3, P[X_1 = 3] = 5/12. \)

(a) (10 points) Calculate
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I \{ X_i \geq 2 \}
\]
and provide justification for the existence of the limit.

(b) (10 points) Specify the distribution of the random variable \( Y \), defined as
\[
Y = \sum_{i=1}^{7} I \{ X_i \leq 1 \}.
\]
6. Suppose that $X_N = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$ P = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} \\
\frac{4}{5} & \frac{1}{5}
\end{pmatrix}. $$

(a) (12 points) Compute the stationary distribution of $X_N$.

(b) (4 points) Suppose that $X_0$ has the distribution obtained in part (a). $P[X_3 = 1] =$?
7. (18 points) Suppose that $X_n$ is the amount of inventory in a store at the end of the time period $n$, and that $D_n$ is the amount of demand that arrives during period $n$. Suppose that $D_1, D_2, \ldots, D_n, \ldots$ is an i.i.d. sequence (independent of $X_0$) of non-negative integer-valued random variables, each with probability mass function $q(\cdot)$; i.e.,
\[
    P[D_1 = i] = q(i), \quad i = 0, 1, 2, \ldots
\]

At the start of each time period, we receive a shipment of 5 units of inventory to the store. Demand that cannot be met is assumed to go away unsatisfied. Under the preceding assumptions, $X_1, X_2, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$. For $n \geq 0$, the inventory level at the end of period $n + 1$ is determined by
\[
    X_{n+1} = \max\{0, X_n + 5 - D_{n+1}\}.
\]
Notice that $\max\{0, a\} = a$ if $a \geq 0$, and $\max\{0, a\} = 0$ if $a < 0$. Hence, the above expression enforces the physical constraint that $X_{n+1} \geq 0$ always, and $X_{n+1} = 0$ when the demand $D_{n+1}$ exceeds the available inventory $X_n + 5$.

(a) (6 points) Determine the transition probabilities $P_{ij}$ for all $i \geq 0$ and $j = 1, \ldots, i + 5$.

(b) (6 points) Determine the transition probabilities $P_{ij}$ for all $i \geq 0$ and $j \geq i + 6$. 

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(c) (6 points) Determine the transition probabilities $P_{i0}$ for all $i \geq 0$. 