Introduction to Random Processes

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Introductions

Class description and contents

Gambling
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Where? We meet in Goergen Hall 109
When? Mondays and Wednesdays 4:50 pm to 6:05 pm
My office hours, Tuesdays at 10 am
  Anytime, as long as you have something interesting to tell me

Class website
http://www.ece.rochester.edu/~gmateosb/ECE440.html
Teaching assistants

- Three great TAs to help you with your homework

- **Yang Li**
  - Hopeman 412, yli131@ur.rochester.edu
  - His office hours, **Mondays at 3 pm**

- **Rasoul Shafipour**
  - Hopeman 412, rshafipo@ur.rochester.edu
  - His office hours, **Thursdays at 3 pm**

- **Chang Ye**
  - Hopeman 414, cye7@ur.rochester.edu
  - His office hours, **Fridays at 1 pm**
Prerequisites

(I) Probability theory
- Random (Stochastic) processes are collections of random variables
- Basic knowledge expected. Will review in the first five lectures

(II) Calculus and linear algebra
- Integrals, limits, infinite series, differential equations
- Vector/matrix notation, systems of linear equations, eigenvalues

(III) Programming in Matlab
- Needed for homework
- If you know programming you can learn Matlab in one afternoon
  ⇒ But it has to be this afternoon
Homework and grading

(I) Homework sets (10 in 14 weeks) worth 28 points
- Important and demanding part of this class
- Collaboration accepted, welcomed, and encouraged

(II) Midterm examination on Monday November 6 worth 36 points

(III) Final take-home examination on December 10-13 worth 36 points
- Work independently. This time no collaboration, no discussion
- ECE 271 students get 10 free points
- At least 60 points are required for passing (C grade)
- B requires at least 75 points. A at least 92. No curve
  ⇒ Goal is for everyone to earn an A
Textbooks

▶ Textbook for the class is

⇒ Available online: http://www.library.rochester.edu/

▶ Also good for topics including Markov chains, queuing models is


▶ Both on reserve for the class in Carlson Library
Be nice

- I work hard for this course, expect you to do the same
  - Come to class, be on time, pay attention, ask
  - Do all of your homework
  - Do not hand in as yours the solution of others (or mine)
  - Do not collaborate in the take-home final

- A little bit of (conditional) probability ...
- Probability of getting an E in this class is 0.04
- Probability of getting an E given you skip 4 homework sets is 0.7
  ⇒ I’ll give you three notices, afterwards, I’ll give up on you

- Come and learn. Useful down the road
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Gambling
Stochastic systems

- **Stochastic system**: Anything random that evolves in time
  - Time can be discrete $n = 0, 1, 2 \ldots$, or continuous $t \in [0, \infty)$

- More formally, random processes assign a function to a random event
- Compare with “random variable assigns a value to a random event”

- Can interpret a random process as a collection of random variables
  - Generalizes concept of random vector to functions
  - Or generalizes the concept of function to random settings
A voice recognition system

- Random event \( \sim \) word spoken. Random process \( \sim \) the waveform
- Try the file speech_signals.m

“Hi”

“Good”

“Bye”

‘S’
(I) **Probability theory review** (5 lectures)
- Probability spaces, random variables, independence, expectation
- Conditional probability: time $n + 1$ given time $n$, future given past ...
- Limits in probability, almost sure limits: behavior as $n \to \infty$ ...
- Common probability distributions (binomial, exponential, Poisson, Gaussian)

- Random processes are complicated entities
  - ⇒ Restrict attention to particular classes that are somewhat tractable

(II) **Markov chains** (7 lectures)

(III) **Continuous-time Markov chains** (8 lectures)

(IV) **Stationary random processes** (5 lectures)

- Midterm covers up to Markov chains
Markov chains

- **Countable set of states** 1, 2, ... At discrete time $n$, state is $X_n$
- **Memoryless (Markov) property**
  - Probability of next state $X_{n+1}$ depends on current state $X_n$
  - But not on past states $X_{n-1}, X_{n-2}, ...$
- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Tomorrow’s mood only affected by today’s mood
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so
- **Of interest:** classification of states, ergodicity, limiting distributions
- **Applications:** Google’s page rank, epidemic modeling, queues, ...

![Diagram of Markov chain with states H and S, transition probabilities: 0.8 from H to H, 0.3 from S to S, 0.2 from H to S, 0.7 from S to H]
Continuous-time Markov chains

- **Countable** set of states 1, 2, …. **Continuous-time** index $t$, state $X(t)$
  - Transition between states can happen at any time
  - **Markov**: Future independent of the past given the present

- Probability of changing state in an infinitesimal time $dt$

- **Of interest**: Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions

- **Applications**: Chemical reactions, queues, communication networks, weather forecasting, …
Stationary random processes

- **Continuous** time \( t \), **continuous state** \( X(t) \), not necessarily Markov
- Prob. distribution of \( X(t) \) constant or becomes constant as \( t \) grows
  \[\Rightarrow\] System has a **steady state in a random sense**

- **Of interest:** Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density

- **Applications:** Black Scholes model for option pricing, radar, speech recognition, noise in electric circuits, filtering and equalization ...
Gambling

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Gambling
An interesting betting game

- There is a certain game in a certain casino in which ...
  - ⇒ Your chances of winning are $p > 1/2$
- You place $1$ bets
  - (a) With probability $p$ you gain $1$; and
  - (b) With probability $1 - p$ you lose your $1$ bet
- The catch is that you either
  - (a) Play until you go broke (lose all your money)
  - (b) Keep playing forever
- You start with an initial wealth of $w_0$
- Q: Shall you play this game?
Let $t$ be a time index (number of bets placed)

- Denote as $X(t)$ the outcome of the bet at time $t$:
  - $X(t) = 1$ if bet is won (w.p. $p$)
  - $X(t) = 0$ if bet is lost (w.p. $1 - p$)

- $X(t)$ is called a Bernoulli random variable with parameter $p$

- Denote as $W(t)$ the player’s wealth at time $t$. Initialize $W(0) = w_0$

- At times $t > 0$ wealth $W(t)$ depends on past wins and losses:
  - When bet is won $W(t + 1) = W(t) + 1$
  - When bet is lost $W(t + 1) = W(t) - 1$

- More compactly can write $W(t + 1) = W(t) + (2X(t) - 1)$
  - Only holds so long as $W(t) > 0$
\begin{verbatim}
t = 0; w(t) = w_0; \text{max}_t = 10^3; // Initialize variables
% repeat while not broke up to time \text{max}_t
\textbf{while} (w(t) > 0) & (t < \text{max}_t) \textbf{do}
  x(t) = \text{random('bino',1,p)}; % Draw Bernoulli random variable
  \textbf{if} x(t) == 1 \textbf{then}
    w(t + 1) = w(t) + b; % If x = 1 wealth increases by b
  \textbf{else}
    w(t + 1) = w(t) - b; % If x = 0 wealth decreases by b
\textbf{end}
t = t + 1;
\textbf{end}
\end{verbatim}

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$

- Q: Shall we play?
One lucky player

- She didn’t go broke. After $t = 1000$ bets, her wealth is $W(t) = 109$
  - Less likely to go broke now because wealth increased
Two lucky players

- After $t = 1000$ bets, wealths are $W_1(t) = 109$ and $W_2(t) = 139$

  $\Rightarrow$ Increasing wealth seems to be a pattern
Ten lucky players

- Weights $W_j(t)$ after $t = 1000$ bets between 78 and 139

$\Rightarrow$ Increasing wealth is definitely a pattern
One unlucky player

- But this does not mean that all players will turn out as winners
  ⇒ The twelfth player $j = 12$ goes broke
One unlucky player

But this does not mean that all players will turn out as winners

⇒ The twelfth player $j = 12$ goes broke
One hundred players

- All players (except for $j = 12$) end up with substantially more money
It is not difficult to find a line estimating the average of $W(t)$

$$\tilde{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t \quad (\text{recall } p = 0.55)$$
Where does the average tendency come from?

- Assuming we do not go broke, we can write
  \[ W(t + 1) = W(t) + \left( 2X(t) - 1 \right), \quad t = 0, 1, 2, \ldots \]

- The assumption is incorrect as we saw, but suffices for simplicity

- Taking expectations on both sides and using linearity of expectation
  \[ \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + \left( 2\mathbb{E}[X(t)] - 1 \right) \]

- The expected value of Bernoulli \( X(t) \) is
  \[ \mathbb{E}[X(t)] = 1 \times P(X(t) = 1) + 0 \times P(X(t) = 0) = p \]

- Which yields
  \[ \mathbb{E}[W(t + 1)] = \mathbb{E}[W(t)] + (2p - 1) \]

- Applying recursively
  \[ \Rightarrow \mathbb{E}[W(t + 1)] = w_0 + (2p - 1)(t + 1) \]
Gambling as LTI system with stochastic input

- Recall the evolution of wealth $W(t + 1) = W(t) + \left(2X(t) - 1\right)$

- View $W(t + 1)$ as output of LTI system with random input $2X(t) - 1$

- Recognize accumulator $\Rightarrow W(t + 1) = w_0 + \sum_{\tau=0}^{t} \left(2X(\tau) - 1\right)$
  - Useful, a lot we can say about sums of random variables

- Filtering random processes in signal processing, communications, …
Numerical analysis of simulation outcomes

- For a more accurate approximation, analyze simulation outcomes.
- Consider $J$ experiments. Each yields a wealth history $W_j(t)$.
- Can estimate the average outcome via the sample average $\bar{W}_J(t)$.

$$\bar{W}_J(t) := \frac{1}{J} \sum_{j=1}^{J} W_j(t)$$

- Do not confuse $\bar{W}_J(t)$ with $\mathbb{E}[W(t)]$.
  - $\bar{W}_J(t)$ is computed from experiments, it is a random quantity in itself.
  - $\mathbb{E}[W(t)]$ is a property of the random variable $W(t)$.
  - We will see later that for large $J$, $\bar{W}_J(t) \rightarrow \mathbb{E}[W(t)]$. 
Analysis of simulation outcomes: mean

- Expected value $\mathbb{E}[W(t)]$ in black
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)
There is more information in the simulation’s output

Estimate the probability distribution function (pdf) \( \Rightarrow \) Histogram

Consider a set of points \( w^{(0)}, \ldots, w^{(N)} \)

Indicator function of the event \( w^{(n)} \leq W_j(t) < w^{(n+1)} \)

\[
\mathbb{I}\left\{ w^{(n)} \leq W_j(t) < w^{(n+1)} \right\} = \begin{cases} 
1, & \text{if } w^{(n)} \leq W_j(t) < w^{(n+1)} \\
0, & \text{otherwise}
\end{cases}
\]

Histogram is then defined as

\[
H\left[ t; w^{(n)}, w^{(n+1)} \right] = \frac{1}{J} \sum_{j=1}^{J} \mathbb{I}\left\{ w^{(n)} \leq W_j(t) < w^{(n+1)} \right\}
\]

Fraction of experiments with wealth \( W_j(t) \) between \( w^{(n)} \) and \( w^{(n+1)} \)
The pdf broadens and shifts to the right \((t = 10, 50, 100, 200)\)
What is this class about?

- Analysis and simulation of **stochastic systems**
  - A system that **evolves in time** with some **randomness**

- They are usually quite **complex** ⇒ Simulations

- We will learn how to **model** stochastic systems, e.g.,
  - \( X(t) \) Bernoulli with parameter \( p \)
  - \( W(t + 1) = W(t) + 1, \) when \( X(t) = 1 \)
  - \( W(t + 1) = W(t) - 1, \) when \( X(t) = 0 \)

- ... how to **analyze** their properties, e.g., \( \mathbb{E}[W(t)] = w_0 + (2p - 1)t \)

- ... and how to **interpret** simulations and experiments, e.g.,
  - Average tendency through sample average
  - Estimate probability distributions via histograms