Arbitrages and pricing of stock options

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Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Arbitrage

- Bet on different events with each outcome paying a random return

- **Arbitrage**: possibility of devising a betting strategy that
  - Guarantees a positive return
  - No matter the combined outcome of the events

- Arbitrages often involve operating in two (or more) different markets
Sports betting example

Ex: Booker 1  ⇒ Yankees win pays 1.5:1, Yankees loss pays 3:1

- Bet $x$ on Yankees and $y$ against Yankees. Guaranteed earnings?

  Yankees win: $0.5x - y > 0 \Rightarrow x > 2y$
  Yankees loose: $-x + 2y > 0 \Rightarrow x < 2y$

  ⇒ Arbitrage not possible. Notice that $1/(1.5) + 1/3 = 1$

Ex: Booker 2  ⇒ Yankees win pays 1.4:1, Yankees loss pays 3.1:1

- Bet $x$ on Yankees and $y$ against Yankees. Guaranteed earnings?

  Yankees win: $0.4x - y > 0 \Rightarrow x > 2.5y$
  Yankees loose: $-x + 2.1y > 0 \Rightarrow x < 2.1y$

  ⇒ Arbitrage not possible. Notice that $1/(1.4) + 1/(3.1) > 1$
Sports betting example (continued)

- First condition on Booker 1 and second on Booker 2 are compatible

- Bet $x$ on Yankees on Booker 1, $y$ against Yankees on Booker 2

- Guaranteed earnings possible. Make e.g., $x = 2066$, $y = 1000$

  Yankees win: $0.5 \times 2066 - 1000 = 33$
  Yankees loose: $-2066 + 2.1 \times 1000 = 34$

  ⇒ Arbitrage possible. Notice that $1/(1.5) + 1/(3.1) < 1$

- Sport bookies coordinate their odds to avoid arbitrage opportunities
  ⇒ Like card counting in casinos, arbitrage betting not illegal
  ⇒ But you will be banned if caught involved in such practices

- If you plan on doing this, do it on, e.g., currency exchange markets
Let events on which bets are posted be \( k = 1, 2, \ldots, K \)

Let \( j = 1, 2, \ldots, J \) index possible joint outcomes

- Joint realizations, also called “world realization”, or “world outcome”

If world outcome is \( j \), event \( k \) yields return \( r_{jk} \) per unit invested (bet)

Invest (bet) \( x_k \) in event \( k \) \( \Rightarrow \) return for world \( j \) is \( x_k r_{jk} \)

\( \Rightarrow \) Bets \( x_k \) can be positive \( (x_k > 0) \) or negative \( (x_k < 0) \)

\( \Rightarrow \) Positive = regular bet (buy). Negative = short bet (sell)

Total earnings \( \Rightarrow \sum_{k=1}^{K} x_k r_{jk} = x^T r_j \)

Vectors of returns for outcome \( j \) \( \Rightarrow r_j := [r_{j1}, \ldots, r_{jK}]^T \) (given)

Vector of bets \( \Rightarrow x := [x_1, \ldots, x_K]^T \) (controlled by gambler)
Ex: Booker 1 $\Rightarrow$ Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are $K = 2$ events to bet on
  $\Rightarrow$ A Yankees’ win ($k = 1$) and a Yankees’ loss ($k = 2$)
- Naturally, there are $J = 2$ possible outcomes
  $\Rightarrow$ Yankees won ($j = 1$) and Yankess lost ($j = 2$)
- Q: What are the returns?

  Yankees win ($j = 1$): $r_{11} = 0.5, \quad r_{12} = -1$
  Yankees loose ($j = 2$): $r_{21} = -1, \quad r_{22} = 2$

  $\Rightarrow$ Return vectors are thus $r_1 = [0.5, -1]^T$ and $r_2 = [-1, 2]^T$
- Bet $x$ on Yankees and $y$ against Yankees, vector of bets $x = [x, y]^T$
Arbitrage (clearly defined now)

- Arbitrage is possible if there exists investment strategy $x$ such that
  $$x^T r_j > 0, \quad \text{for all } j = 1, \ldots, J$$

- Equivalently, arbitrage is possible if
  $$\max_x \left( \min_j (x^T r_j) \right) > 0$$

- Earnings $x^T r_j$ are the inner product of $x$ and $r_j$ (i.e., $\perp$ projection)

  $\Rightarrow$ Positive earnings if angle between $x$ and $r_j < \pi/2$ ($90^\circ$)
When is arbitrage possible?

- There is a line that leaves all $r_j$ vectors to one side
  - Arbitrage possible
  - Prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
    \[ E_p(r) = \sum_{j=1}^{J} p_j r_j = 0 \]
    does not exist

- There is not a line that leaves all $r_j$ vectors to one side
  - Arbitrage not possible
  - There is prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that
    \[ E_p(r) = \sum_{j=1}^{J} p_j r_j = 0 \]
  - Think of $p_j$ as scaling factors
Arbitrage theorem

- Have demonstrated the following result, called arbitrage theorem

  ⇒ Formal proof follows from duality theory in optimization

**Theorem**

*Given vectors of returns* $r_j \in \mathbb{R}^K$ *associated with random world outcomes* $j = 1, \ldots, J$, *an arbitrage is not possible if and only if there exists a probability vector* $p = [p_1, \ldots, p_J]^T$ with $p_j \geq 0$ and $p^T 1 = 1$, *such that* $\mathbb{E}_p(r) = 0$. Equivalently,

$$\max_x \left( \min_j (x^T r_j) \right) \leq 0 \iff \sum_{j=1}^J p_j r_j = 0$$

- Prob. vector $p$ is **NOT** the prob. distribution of events $j = 1, \ldots, J$
Example: Arbitrages in sports betting

Ex: Booker 1 ⇒ Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are \( K = 2 \) events to bet on, \( J = 2 \) possible outcomes

- Q: What are the returns?

  Yankees win \((j = 1)\): \( r_{11} = 0.5, \quad r_{12} = -1 \)
  Yankees loose \((j = 2)\): \( r_{21} = -1, \quad r_{22} = 2 \)

⇒ Return vectors are thus \( r_1 = [0.5, -1]^T \) and \( r_2 = [-1, 2]^T \)

- Arbitrage impossible if there is \( 0 \leq p \leq 1 \) such that

\[
\mathbb{E}_p(r) = p \times \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} + (1 - p) \times \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0
\]

⇒ Straightforward to check that \( p = 2/3 \) satisfies the equation
Consider a stock price $X(nh)$ that follows a geometric random walk

$$X((n+1)h) = X(nh)e^{\sigma \sqrt{h} Y_n}$$

$Y_n$ is a binary random variable with probability distribution

$$P(Y_n = 1) = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{h} \right), \quad P(Y_n = -1) = \frac{1}{2} \left( 1 - \frac{\mu}{\sigma} \sqrt{h} \right)$$

$\Rightarrow$ As $h \to 0$, $X(nh)$ becomes geometric Brownian motion

$\textbf{Q:}$ Are there arbitrage opportunities in trading this stock?

$\Rightarrow$ Too general, let us consider a narrower problem
Consider the following investment strategy (stock flip):

**Buy:** Buy $1 in stock at time 0 for price $X(0)$ per unit of stock

**Sell:** Sell stock at time $h$ for price $X(h)$ per unit of stock

Cost of transaction is $1$. Units of stock purchased are $1/X(0)$

- Cash after selling stock is $X(h)/X(0)$
- Return on investment is $X(h)/X(0) - 1$

There are two possible outcomes for the price of the stock at time $h$

- May have $Y_0 = 1$ or $Y_0 = -1$ respectively yielding

$$X(h) = X(0)e^{\sigma \sqrt{h}}$$

Possible returns are therefore

$$r_1 = \frac{X(0)e^{\sigma \sqrt{h}}}{X(0)} - 1 = e^{\sigma \sqrt{h}} - 1$$

$$r_2 = \frac{X(0)e^{-\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\sigma \sqrt{h}} - 1$$
Present value of returns

- One dollar at time $h$ is not the same as 1 dollar at time 0
  $\Rightarrow$ Must take into account the time value of money

- Interest rate of a risk-free investment is $\alpha$ continuously compounded
  $\Rightarrow$ In practice, $\alpha$ is the money-market rate (time value of money)

- Prices have to be compared at their present value

- The present value (at time 0) of $X(h)$ is $X(h)e^{-\alpha h}$
  $\Rightarrow$ Return on investment is $e^{-\alpha h}X(h)/X(0) - 1$

- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are

  
  $$r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{\sigma \sqrt{h}} - 1,$$

  $$r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma \sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma \sqrt{h}} - 1.$$

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Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1 - q)r_2 = 0$$

$\Rightarrow$ Arbitrage theorem in one dimension (only one bet, stock flip)

Substituting $r_1$ and $r_2$ for their respective values

$$q \left( e^{-\alpha h} e^{\sigma \sqrt{h}} - 1 \right) + (1 - q) \left( e^{-\alpha h} e^{-\sigma \sqrt{h}} - 1 \right) = 0$$

Can be easily solved for $q$. Expanding product and reordering terms

$$qe^{-\alpha h} e^{\sigma \sqrt{h}} + (1 - q)e^{-\alpha h} e^{-\sigma \sqrt{h}} = 1$$

Multiplying by $e^{\alpha h}$ and grouping terms with a $q$ factor

$$q \left( e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}} \right) = e^{\alpha h} - e^{-\sigma \sqrt{h}}$$
Solving for $q$ finally yields

$$q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}}$$

For small $h$ we have $e^{\alpha h} \approx 1 + \alpha h$ and $e^{\pm \sigma \sqrt{h}} \approx 1 \pm \sigma \sqrt{h} + \sigma^2 h / 2$

Thus, the value of $q$ as $h \to 0$ may be approximated as

$$q \approx \frac{1 + \alpha h - \left(1 - \sigma \sqrt{h} + \sigma^2 h / 2\right)}{1 + \sigma \sqrt{h} - \left(1 - \sigma \sqrt{h}\right)} = \frac{\sigma \sqrt{h} + (\alpha - \sigma^2 / 2) h}{2\sigma \sqrt{h}}$$

$$= \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2 / 2}{\sigma} \sqrt{h}\right)$$

Approximation proves that at least for small $h$, then $0 < q < 1$

$\Rightarrow$ **Arbitrage not possible**

Also, suspiciously similar to probabilities of geometric random walk

$\Rightarrow$ **Key observation as we’ll see next**
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Stock prices $X(nh)$ follow geometric random walk (drift $\mu$, variance $\sigma^2$) \[\Rightarrow\] Risk-free investment has return $\alpha$ (time value of money)

Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

\[ q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}} \]

Notice that $q$ satisfies the equation (which we’ll use later on)

\[ q e^{\sigma \sqrt{h}} + (1 - q) e^{-\sigma \sqrt{h}} = e^{\alpha h} \]

Q: Can we have arbitrage using a more complex set of possible bets?
Consider the following general investment strategy:

**Observe:** Observe the stock price at times $h, 2h, \ldots, nh$

**Compare:** Is $X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n$?

**Buy:** If above answer is yes, buy stock at price $X(nh)$

**Sell:** Sell stock at time $mh$ ($m > n$) for price $X(mh)$

Possible bets are the observed values of the stock $x_1, x_2, \ldots, x_n$

$\Rightarrow$ There are $2^n$ possible bets

Possible outcomes are value at time $mh$ and observed values

$\Rightarrow$ There are $2^m$ possible outcomes
Explanation of general investment strategy

- There are $2^n$ possible bets:
  - Bet 1 = $n$ price increases in $1, \ldots, n$
  - Bet 2 = price increases in $1, \ldots, n-1$ and price decrease in $n$
  - ...

- For each bet we have $2^{m-n}$ possible outcomes:
  - $m-n$ price increases in $n+1, \ldots, m$
  - Price increases in $n+1, \ldots, m-1$ and price decrease in $m$
  - ...

<table>
<thead>
<tr>
<th>Bet 1</th>
<th>$X(h)$</th>
<th>$X(2h)$</th>
<th>$X(3h)$</th>
<th>$X(nh)$</th>
<th>$X((n+1)h)$</th>
<th>$X((n+2)h)$</th>
<th>$X(mh)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^\sigma \sqrt{h}$</td>
<td>$e^{2\sigma \sqrt{h}}$</td>
<td>$e^{3\sigma \sqrt{h}}$</td>
<td>$e^{n\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{m\sigma \sqrt{h}}$</td>
<td></td>
</tr>
<tr>
<td>bet 2</td>
<td>$e^\sigma \sqrt{h}$</td>
<td>$e^{2\sigma \sqrt{h}}$</td>
<td>$e^{3\sigma \sqrt{h}}$</td>
<td>$e^{(n-2)\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{(m-2)\sigma \sqrt{h}}$</td>
</tr>
<tr>
<td>bet $2^n$</td>
<td>$e^{-\sigma \sqrt{h}}$</td>
<td>$e^{-2\sigma \sqrt{h}}$</td>
<td>$e^{-3\sigma \sqrt{h}}$</td>
<td>$e^{-n\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-2\sigma \sqrt{h}}$</td>
<td>$X(nh)e^{-m\sigma \sqrt{h}}$</td>
</tr>
</tbody>
</table>

- Table assumes $X(0) = 1$ for simplicity

outcomes per each bet
Define the prob. distribution $q$ over possible outcomes as follows:

Start with a sequence of i.i.d. binary RVs $Y_n$, probabilities

$$P(Y_n = 1) = q, \quad P(Y_n = -1) = 1 - q$$

⇒ With $q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}}$ as in slide 18

Joint prob. distribution $q$ on $X(h), X(2h), \ldots, X(mh)$ from

$$X((n+1)h) = X(nh)e^{\sigma \sqrt{h}Y_n}$$

⇒ Recall this is NOT the prob. distribution of $X(nh)$

Will show that expected value of earnings with respect to $q$ is null

⇒ By arbitrage theorem, arbitrages are not possible
Consider a time 0 unit investment in given arbitrary outcome

Stock units purchased depend on the price $X(nh)$ at buying time

$$\text{Units bought} = \frac{1}{X(nh)e^{-\alpha nh}}$$

⇒ Have corrected $X(nh)$ to express it in time 0 values

Cash after selling stock given by price $X(mh)$ at sell time $m$

$$\text{Cash after sell} = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}$$

⇒ Return is then

$$r(X(h), \ldots, X(mh)) = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1$$

⇒ Depends on $X(mh)$ and $X(nh)$ only
Expected return with respect to measure $q$

- Expected value of all possible returns with respect to $q$ is

$$
\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]
$$

- Condition on observed values $X(h), \ldots, X(nh)$

$$
\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ \mathbb{E}_{q(n+1:m)} \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \mid X(h), \ldots, X(nh) \right] \right]
$$

- In innermost expectation $X(nh)$ is given. Furthermore, process $X$ is Markov, so conditioning on $X(h), \ldots, X((n-1)h)$ is irrelevant. Thus

$$
\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_{q(1:n)} \left[ \frac{\mathbb{E}_{q(n+1:m)} \left[ X(mh) \mid X(nh) \right] e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]
$$
Expected value of future values (measure $q$)

- Need to find expectation of future value $\mathbb{E}_{q(n+1:m)} [X(mh) \mid X(nh)]$

- From recursive relation for $X(nh)$ in terms of $Y_n$ sequence

$$X(mh) = X((m-1)h) e^{\sigma \sqrt{h} Y_{m-1}}$$
$$= X((m-2)h) e^{\sigma \sqrt{h} Y_{m-1}} e^{\sigma \sqrt{h} Y_{m-2}}$$
$$\vdots$$
$$= X(nh) e^{\sigma \sqrt{h} Y_{m-1}} e^{\sigma \sqrt{h} Y_{m-2}} \ldots e^{\sigma \sqrt{h} Y_n}$$

- All the $Y_n$ are independent. Then, upon taking expectations

$$\mathbb{E}_{q(n+1:m)} [X(mh) \mid X(nh)] = X(nh) \mathbb{E} \left[ e^{\sigma \sqrt{h} Y_{m-1}} \right] \mathbb{E} \left[ e^{\sigma \sqrt{h} Y_{m-2}} \right] \ldots \mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$$

- Need to determine expectation of relative price change $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$
Expectation of relative price change (measure $q$)

- The expected value of the relative price change $E \left[ e^{\sigma \sqrt{h} Y_n} \right]$ is

$$E \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} \Pr \left[ Y_n = 1 \right] + e^{-\sigma \sqrt{h}} \Pr \left[ Y_n = -1 \right]$$

- According to definition of measure $q$, it holds

$$\Pr \left[ Y_n = 1 \right] = q, \quad \Pr \left[ Y_n = -1 \right] = 1 - q$$

- Substituting in expression for $E \left[ e^{\sigma \sqrt{h} Y_n} \right]$:

$$E \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} q + e^{-\sigma \sqrt{h}} (1 - q) = e^{\alpha h}$$

$\Rightarrow$ Follows from definition of probability $q$ [cf. slide 18]

- Reweave the quilt:
  (i) Use expected relative price change to compute expected future value
  (ii) Use expected future value to obtain desired expected return
Reweave the quilt

- Plug $E\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\alpha h}$ into expression for expected future value

$$E_{q(n+1:m)}[X(mh) | X(nh)] = X(nh)e^{\alpha h}e^{\alpha h} \ldots e^{\alpha h} = X(nh)e^{\alpha(m-n)h}$$

- Substitute result into expression for expected return

$$E_q\left[r(X(h), \ldots, X(mh))\right] = E_{q(1:n)}\left[\frac{X(nh)e^{\alpha(m-n)h}e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

- Exponentials cancel out, finally yielding

$$E_q\left[r(X(h), \ldots, X(mh))\right] = E_{q(1:n)}[1 - 1] = 0$$

$\Rightarrow$ Arbitrage not possible if $0 \leq q \leq 1$ exists (true for small $h$)
What if prices follow a geometric Brownian motion?

- Suppose stock prices follow a geometric Brownian motion, i.e.,
  \[ X(t) = X(0)e^{Y(t)} \]
  \[ \Rightarrow Y(t) \text{ Brownian motion with drift } \mu \text{ and variance } \sigma^2 \]

- Q: What is the no arbitrage condition?

- Approximate geometric Brownian motion by geometric random walk
  \[ \Rightarrow \text{Approximation arbitrarily accurate by letting } h \to 0 \]

- No arbitrage measure \( q \) exists for geometric random walk
  - This requires \( h \) sufficiently small
  - Notice that prob. distribution \( q = q(h) \) is a function of \( h \)

- Existence of the prob. distribution \( q := \lim_{h \to 0} q(h) \) proves that
  \[ \Rightarrow \text{Arbitrages are not possible in stock trading} \]
Recall that as $h \to 0 \Rightarrow q \approx \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right)$

$\Rightarrow 1 - q \approx \frac{1}{2} \left(1 - \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h}\right)$

Thus, measure $q := \lim_{h \to 0} q(h)$ is a geometric Brownian motion

$\Rightarrow$ Variance $\sigma^2$ (same as stock price)

$\Rightarrow$ Drift $\alpha - \sigma^2/2$

Measure showing arbitrage impossible a geometric Brownian motion

$\Rightarrow$ Which is also the way stock prices evolve as $h \to 0$

Furthermore, the variance is the same as that of stock prices

$\Rightarrow$ Different drifts $\Rightarrow \mu$ for stocks and $\alpha - \sigma^2/2$ for no arbitrage
Expected investment growth

- Compute expected return on an investment on stock $X(t)$
  - Buy 1 share of stock at time 0. Cash invested is $X(0)$
  - Sell stock at time $t$. Cash after sell is $X(t)$

- Expected value of cash after sell given $X(0)$ is
  
  $$
  E[X(t) \mid X(0)] = X(0)e^{(\mu+\sigma^2/2)t}
  $$

- Alternatively, invest $X(0)$ risk free in the money market
  - Guaranteed cash at time $t$ is $X(0)e^{\alpha t}$

- Invest in stock only if $\mu + \sigma^2/2 > \alpha$ ⇒ “Risk premium” exists
Stock prices follow a geometric Brownian motion $X(t) = X(0)e^{Y(t)}$

$\Rightarrow Y(t)$ Brownian motion with drift $\mu$ and variance $\sigma^2$

Q: What is the expected return $\mathbb{E}[X(t) \mid X(0)]$?

Note first that $\mathbb{E}[X(t) \mid X(0)] = X(0)\mathbb{E}[e^{Y(t)} \mid X(0)]$

Using that $Y(t)$ has independent increments

$$\mathbb{E}[e^{Y(t)} \mid X(0)] = \mathbb{E}[e^{Y(t)}]$$

$\Rightarrow$ Next we focus on computing $\mathbb{E}[e^{Y(t)}]$
Proof of expected return formula (cont.)

Since \( Y(t) \sim N(\mu t, \sigma^2 t) \)

\[
\mathbb{E} \left[ e^{Y(t)} \right] = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^y e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}} dy
\]

Completing the squares in the argument of the exponential we have

\[
y - \frac{(y - \mu t)^2}{2\sigma^2 t} = \frac{-y^2 + 2(\mu + \sigma^2)ty - \mu^2 t^2}{2\sigma^2 t} = \frac{- (y - (\mu + \sigma^2)t)^2}{2\sigma^2 t} + \frac{2\mu\sigma^2 t^2 + \sigma^4 t^2}{2\sigma^2 t}
\]

The blue term does not depend on \( y \), red integral equals 1

\[
\mathbb{E} \left[ e^{Y(t)} \right] = e^{\left(\mu + \frac{\sigma^2}{2}\right)t} \times \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^{-\frac{(y-(\mu+\sigma^2)t)^2}{2\sigma^2 t}} dy = e^{\left(\mu + \frac{\sigma^2}{2}\right)t}
\]

Putting the pieces together, we obtain

\[
\mathbb{E} \left[ X(t) \mid X(0) \right] = X(0) \mathbb{E} \left[ e^{Y(t)} \right] = X(0)e^{(\mu+\sigma^2/2)t}
\]
Risk neutral measure

- Compute expected return as if $q$ were the actual distribution
  - Recall that $q$ is **NOT** the actual distribution
  - As before, cash invested is $X(0)$ and cash after sale is $X(t)$

- Expected cash value is different because prob. distribution is different
  \[
  E_q [X(t) \mid X(0)] = X(0)e^{(\alpha - \sigma^2/2 + \sigma^2/2)t} = X(0)e^{\alpha t}
  \]
  - Same return as risk-free investment regardless of parameters

- Measure $q$ is called **risk neutral measure**
  - Risky stock investments yield same return as risk-free one
  - “Alternate universe”, investors do not demand risk premiums

- Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)
  - Basis for Black-Scholes formula for option pricing
Martingale as basis for fair pricing

- A continuous-time process \( X(t) \) is a **martingale** if for \( t, s \geq 0 \)

\[
\mathbb{E} \left[ X(t + s) \mid X(u), 0 \leq u \leq t \right] = X(t)
\]

\( \Rightarrow \) Expected future value = present value, even given process history

- Model of a fair, e.g., gambling game. **Excludes winning strategies**

\( \Rightarrow \) Even with prior info. of outcomes (cards drawn from the deck)

- For risk-neutral measure \( q \), time 0 prices \( e^{-\alpha t}X(t) \) form a martingale

\[
\mathbb{E}_q \left[ e^{-\alpha(t+s)}X(t + s) \mid e^{-\alpha u}X(u), 0 \leq u \leq t \right] = e^{-\alpha t}X(t)
\]

- **Key principle**: stock price = expected discounted return

\[
X(0) = \mathbb{E}_q \left[ e^{-\alpha t}X(t) \mid X(0) \right]
\]

\( \Rightarrow \) Fair pricing, cannot devise a winning strategy (arbitrage)
Stock prices form a martingale under $q$ (proof)

- Recall measure $q$ is a geometric Brownian motion $X(t) = e^{Y(t)}$
  - Variance $\sigma^2$ (same as stock price)
  - Drift $\alpha - \sigma^2/2$

Proof.

$$
\mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{Y(t+s)} \mid e^{-\alpha u} e^{Y(u)}, 0 \leq u \leq t \right] \\
= \mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{Y(t+s)} \mid e^{-\alpha t} e^{Y(t)} \right] \quad Y(t) \text{ is Markov}
$$

$$
= \mathbb{E}_q \left[ e^{-\alpha(t+s)} e^{[Y(t+s) - Y(t)] + Y(t)} \mid e^{-\alpha t} e^{Y(t)} \right] \quad \text{Add and subtract } Y(t)
$$

$$
= e^{-\alpha t} e^{Y(t)} \mathbb{E}_q \left[ e^{-\alpha s} e^{[Y(t+s) - Y(t)]} \right] \quad \text{Independent increments}
$$

$$
= e^{-\alpha t} X(t) \mathbb{E}_q \left[ e^{-\alpha s} e^{Y(s)} \right] \quad \text{Stationary increments}
$$

$$
= e^{-\alpha t} X(t) \mathbb{E}_q \left[ e^{Y(s)} \right] = e^{(\mu + \sigma^2/2)s} = e^{\alpha s}
$$
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
An option is a contract to buy shares of a stock at a future time

- Strike time \( t \) = Convened time for stock purchase
- Strike price \( K \) = Price at which stock is purchased at strike time

At time \( t \), option holder may decide to

- Buy a stock at strike price \( K \) = exercise the option
- Do not exercise the option

May buy option at time 0 for price \( c \)

Q: How do we determine the option’s worth, i.e., price \( c \) at time 0?
A: Given by the Black-Scholes formula for option pricing
Let $e^{\alpha t}$ be the compounding of a risk-free investment.

Let $X(t)$ be the stock’s price at time $t$.

- Modeled as geometric Brownian motion, drift $\mu$, variance $\sigma^2$.

Risk neutral measure $q$ is also a geometric Brownian motion.

- Drift $\alpha - \sigma^2/2$ and variance $\sigma^2$. 

Stock price model
Return of option investment

- At time $t$, the option’s worth depends on the stock’s price $X(t)$
- If stock’s price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  $\Rightarrow$ Option is worthless (better to buy stock at current price)
- Since had paid $c$ for the option at time 0, lost $c$ on this investment
  $\Rightarrow$ Return on investment is $r = -c$
- If stock’s price larger than strike price $\Rightarrow X(t) > K$
  $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$
- To obtain return express as time 0 values and subtract $c$
  $$r = e^{-\alpha t}(X(t) - K) - c$$
- May combine both in single equation $\Rightarrow r = e^{-\alpha t}(X(t) - K)_+ - c$
  $\Rightarrow (\cdot)_+ := \max(\cdot, 0)$ denotes projection onto positive reals $\mathbb{R}_+$
Option pricing

- Select option price $c$ to prevent arbitrage opportunities

\[ E_q \left[ e^{-\alpha t} (X(t) - K)_+ - c \right] = 0 \]

⇒ Expectation is with respect to risk neutral measure $q$

- From above condition, the no-arbitrage price of the option is

\[ c = e^{-\alpha t} E_q \left[ (X(t) - K)_+ \right] \]

⇒ Source of Black-Scholes formula for option valuation

⇒ Rest of derivation is just evaluating $E_q \left[ (X(t) - K)_+ \right]$

- Same argument used to price any derivative of the stock’s price
Use fact that \( q \) is a geometric Brownian motion

- Let us evaluate \( E_q \left[ (X(t) - K)_+ \right] \) to compute option’s price \( c \)

- Recall \( q \) is a geometric Brownian motion \( \Rightarrow X(t) = X_0 e^{Y(t)} \)
  \( \Rightarrow X_0 = \text{price at time 0} \)
  \( \Rightarrow Y(t) \) BMD, \( \mu \left( = \alpha - \sigma^2/2 \right) \) and variance \( \sigma^2 \)

- Can rewrite no arbitrage condition as
  \[
c = e^{-\alpha t} E_q \left[ (X_0 e^{Y(t)} - K)_+ \right]
  \]

- \( Y(t) \) is a Brownian motion with drift. Thus, \( Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t) \)
  \[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} (X_0 e^{y} - K)_+ e^{-(y-\mu t)^2/(2\sigma^2 t)} \, dy
  \]
Evaluation of the integral

- Note that \((X_0e^{Y(t)} - K)_+ = 0\) for all values \(Y(t) \leq \log(K/X_0)\)

- Because integrand is null for \(Y(t) \leq \log(K/X_0)\) can write

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\log(K/X_0)}^{\infty} (X_0e^y - K) e^{-(y-\mu t)^2/(2\sigma^2 t)} dy
\]

- Change of variables \(z = (y - \mu t)/\sqrt{\sigma^2 t}\). Associated replacements

  - Variable: \(y \Rightarrow \sqrt{\sigma^2 t}z + \mu t\)
  - Differential: \(dy \Rightarrow \sqrt{\sigma^2 t} dz\)
  - Integration limit: \(\log(K/X_0) \Rightarrow a := \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}}\)

- Option price then given by

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \left(X_0e^\sqrt{\sigma^2 tz+\mu t} - K\right) e^{-z^2/2} dz
\]
Separate in two integrals $c = e^{-\alpha t}(l_1 - l_2)$ where

$$l_1 := \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} X_0 e^{\sqrt{\sigma^2 t}z + \mu t} e^{-z^2/2} \, dz$$

$$l_2 := \frac{K}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-z^2/2} \, dz$$

Gaussian $\Phi$ function (ccdf of standard normal RV)

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-z^2/2} \, dz$$

Comparing last two equations we have $l_2 = K\Phi(a)$

Integral $l_1$ requires some more work
Evaluation of the first integral

- Reorder terms in integral $I_1$

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{\sigma^2 t}z + \mu t} e^{-z^2/2} \, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{\sqrt{\sigma^2 tz - z^2/2}} \, dz$$

- The exponent can be written as a square minus a “constant” (no $z$)

$$-\left(z - \sqrt{\sigma^2 t}\right)^2/2 + \sigma^2 t/2 = -z^2/2 + \sqrt{\sigma^2 t}z - \sigma^2 t/2 + \sigma^2 t/2$$

- Substituting the latter into $I_1$ yields

$$I_1 = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2 + \sigma^2 t/2} \, dz = \frac{X_0 e^{\mu t + \sigma^2 t/2}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2} \, dz$$
Evaluation of the first integral (continued)

- Change of variables $u = z - \sqrt{\sigma^2 t}$ \(\Rightarrow\) $du = dz$ and integration limit

\[
a \Rightarrow b := a - \sqrt{\sigma^2 t} = \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}} - \sqrt{\sigma^2 t}
\]

- Implementing change of variables in $I_1$

\[
I_1 = X_0 e^{\mu t + \sigma^2 t/2} \int_{B}^{\infty} e^{-u^2/2} du = X_0 e^{\mu t + \sigma^2 t/2} \Phi(b)
\]

- Putting together results for $I_1$ and $I_2$

\[
c = e^{-\alpha t} (I_1 - I_2) = e^{-\alpha t} X_0 e^{\mu t + \sigma^2 t/2} \Phi(b) - e^{-\alpha t} K \Phi(a)
\]

- For non-arbitrage stock prices (measure $q$) \(\Rightarrow\) $\alpha = \mu + \sigma^2/2$

\(\Rightarrow\) Substitute to obtain Black-Scholes formula
Black-Scholes

- **Black-Scholes formula for option pricing.** Option cost at time 0 is

\[ c = X_0 \Phi(b) - e^{-\alpha t} K \Phi(a) \]

⇒ \( a := \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}} \) and \( b := a - \sqrt{\sigma^2 t} \)

- Note further that \( \mu = \alpha - \sigma^2/2 \). Can then write \( a \) as

\[ a = \frac{\log(K/X_0) - (\alpha - \sigma^2/2) t}{\sqrt{\sigma^2 t}} \]

⇒ \( X_0 = \) stock price at time 0, \( \sigma^2 = \) volatility of stock
⇒ \( K = \) option’s strike price, \( t = \) option’s strike time
⇒ \( \alpha = \) benchmark risk-free rate of return (cost of money)

- **Black-Scholes formula independent of stock’s mean tendency \( \mu \)**
Glossary

- Arbitrage
- Investment strategy
- Bets, events, outcomes
- Returns and earnings
- Arbitrage theorem
- Geometric Brownian motion
- Stock flip
- Time value of money
- Continuously-compounded interest
- Present value
- Risk-free investment
- Expected return
- Risk premium
- Risk neutral measure
- Pricing of derivatives
- Stock option
- Strike time and price
- Option price
- Stock volatility
- Black-Scholes formula