Graph signals and graph-shift operator

- **Graph signal processing - 101**
  - Network as graph G = (V, E): encode pairwise relationships
  - Interest here not in G itself, but in data associated with nodes in V
  - The object of study is a graph signal
  - Ex: Opinion profile, buffer congestion levels, neural activity, epidemic

- **Graph SP: need to broaden classical SP results to graph signals**
  - Motivation and problem formulation
  - BLIND IDENTIFICATION OF GRAPH FILTERS WITH SPARSE INPUTS
  - Our view: GSP well suited to study network processes
  - Assuming a simple graph, we propose a convex algorithm for blind identification of graph filters. Leveraging recent advances in graph signal processing and classical blind deconvolution, we propose a convex algorithm for blind identification of graph filters with sparse inputs. This task amounts to finding the sources and diffusion coefficients that gave rise to an observed network state.

- **Graph signals and graph-shift operator**
  - (Node) graph signals are mappings \( x \colon V \rightarrow \mathbb{R} \)
  - May be represented as a vector \( x \in \mathbb{R}^N \)
  - Graph G is endowed with a graph-shift operator \( S \)

- **Locality of S and frequency-domain representation**
  - \( S \) is a local linear operator
  - Ex: Adjacency A, Degree D, and Laplacian L
  - Time-shift operator when \( S = A \) for G a directed cycle

- **Frequency response of a graph filter**
  - Using \( S = V A V^{-1} \), filter is \( H = \sum_{k} h_k A^k = V \left( \sum_{k} h_k A^k \right) V^{-1} \)

  - Since \( A \) are diagonal, the GFT-IGT can be used to write \( y = Hx \) as \( y = diag(h)k(x) \)

  - Output at frequency \( k \) depends only on input at frequency \( k \)

  - Frequency response of filter \( H = S - \lambda \mathbf{I} \), where \( \lambda \) is Vandermonde

  - Note that GFT for signals \( (V - \lambda S) \) and filters \( (R - \lambda \mathbf{I}) \) are different

- **Diffusion process as graph filters outputs**
  - Q: Upon observing a graph signal \( y \), how was this signal generated?

  - Postulate the following generative model
  - An original (source) signal \( x^0 \)
  - Diffuse via linear graph dynamics \( S x^0 = x^1 = \cdots = x^L \)

  - Observed signal \( y \) is a linear combination of the diffused signals \( x^0 \)

  - View few elements in \( supp(y) = \{ i \mid x_i \neq 0 \} \) as sources or seeds

Motivation and problem formulation

- **Global opinion profile formed by spreading a rumor**
  - What was the rumor? Who started it?
  - How do people combine the opinions heard to form their own?

- **Problem: Blind identification of graph filters with sparse inputs**
  - Generalizes classic blind deconvolution to graphs

- **Blind identification of graph filters with sparse inputs**
  - Assumes a K-Sparse i.e., \( \|x\|_0 = K \leq S \)

**“Lifting the bilinear inverse problem”**

- **Leverage the frequency response of graph filters (U = V^{-1})**
  - Use \( V \Sigma A V^{-1} \) where \( y = \mathbf{v} \Sigma \mathbf{a}) = \mathbf{u} \)

  - Blind graph filter identification - Non-convex feasibility problem

- **Key observation**: Using the Khatri-Rao product (can write as \( y = \mathbf{v} (\mathbf{w}^T \mathbf{u}) ^{vec(\mathbf{u})} \))

  - \( y = A \mathbf{v} \mathbf{u} \) is a linear combination of the entries of \( Z \) and \( \mathbf{v} \mathbf{u} \)

  - \( Z \) is of rank 1 and row-sparse

  - \( \mathbf{Z} \) is an S-sparse matrix

  - \( \|x\|_0 = K \leq S \)

  - Blind rank minimization achieves perfect recovery when \( N \geq (L - S) \)

  - \( \|x\|_0 = K \leq S \)

- **Rank minimization s. to row-cardinality constraint is NP-hard. Relax!**

  - Define the rank-one matrices \( \mathbf{Z} = \mathbf{A} \mathbf{v} \mathbf{u} \)

  - \( \mathbf{Z} \) is rank-sparse

  - Shrinkage to row-sparse

**Algorithmic approach via convex relaxation**

- **Rank minimization s. to row-cardinality constraint is NP-hard. Relax!**

- **Convex relaxation**
  - Minimize with coefficients \( \mathbf{Z} \) for \( \| \mathbf{Z} \|_1 \) per row and permutation \( k \)

- **Multiple output signals**

  - Leverage multiple output signals to aid the blind identification task

  - We have access to a collection of output signals \( \{y_i\} \)

  - Corresponding to different sparse inputs \( x_i \) but a common filter \( H \)

  - Consider the stacked vectors \( y = [y_1, \ldots, y_N]^T \) and \( s = [s_1, \ldots, s_M]^T \)

  - Define the rank-one matrices \( \mathbf{Z} = \mathbf{A} \mathbf{v} \mathbf{u} \)

  - Solve the row-sparse rank-

- **Numerical tests: Known support, random graph models**

  - Performance in Erdős-Rényi and scale-free graphs of varying size

  - Assume known \( \|x\|_0 = K \geq 0 \)

  - Error quantified as \( \| \mathbf{Z} \|_1 \)

  - Two settings: \( L = 1, 2, 20 \) and \( L = 5, S = 40 \)

  - Rank minimization achieves perfect recovery when \( N \geq (L - S) \)

  - Well-below \( N \geq (L - S) \) needed for least squares to succeed

  - Rank minimization is more robust to the type of graph

**Recovery rate in random graphs: unknown support**

- **Recovery rates on Erdős-Rényi graphs**

  - \( N \) = 50 for varying \( L \) and \( S \)

  - \( P = 1 \) (left), \( P = 1 + \text{reweighted } c_{ij} \) (middle), \( P = 5 + \text{reweighted } c_{ij} \) (right)

- **Exact recovery over non-trivial (L, S) region**

  - Iteratively-reweighted optimization markedly improves recovery

  - Multiple outputs further increases recovery success

- **Performance comparison with alternative methods**

  - Human brain graph of \( N = 66 \) brain regions, \( L = 6 \) and \( S = 6 \)

  - Scalable algorithm using method of multipliers

  - Iteratively-reweighted optimization markedly improves recovery

  - Pseudo-norm \( \|x\|_0 \) needed twice as many observations

**Discussion and road ahead**

- **Identifiability conditions**

  - When is \( \mathbf{y} = \mathbf{x} \) the unique solution (up to scaling)?

  - Deterministic or probabilistic model assumptions

  - Exact recovery conditions

  - When does the convex relaxation succeed?

  - Lower bound on \( N \) to guarantee recovery for given \( L \) and \( S \)

  - Depends on algebraic features of the graph \( S \)

  - Some graphs are more amenable to blind identification that others

  - Unknown shift \( S \)

  - Network topology inference

  - Endogenous application problems

  - Opinion formation in social networks

  - Identity sources of epileptic seizures

  - Event-driven information cascades

  - Trace ‘patient zero’ for an epidemic outbreak

**References**