ECE 272/472. Lecture 2: Sampling

Remember the basic A/D conversion diagram:

\[ X(t) \quad \text{Analog} \quad \text{Low-pass filter} \quad \rightarrow \quad X[n] \quad \text{Sampling} \quad \rightarrow \quad X_a[n] \quad \text{Quantization} \]

Questions:
1. Why do we need the Analog LP filter?
2. How fast should we sample the signal?

(Nyquist - Shannon Sampling Theorem)

Intuitive Explanation:
Consider \( x(t) \) as a complex sinusoidal signal \( x(t) = A e^{j(wt + \phi)} \)

- \( w \): angular frequency: how many radians per second
- \( f \): frequency: how many cycles per second
- 1 cycle = \( 2\pi \) radians

\[ f = \frac{W}{2\pi}, \quad T = \frac{1}{f} = \frac{2\pi}{W} \]

\( \frac{\pi}{2} \) radians have passed

\[ t = \frac{\pi}{2} \cdot \frac{1}{W} \] in this figure.

Sampling: taking snapshots. \( f_s \): Sampling rate/freq. : How many snapshots per second.

\[ T_s = \frac{1}{f_s} \]

Sampling period: \( T_s = \frac{1}{f_s} \)
\[
\frac{f}{f_s} = \frac{\# \text{Cycles}}{1 \text{ sec}} = \frac{1 \text{ sec}}{\# \text{ samples}} = \frac{\# \text{Cycles}}{1 \text{ Sample}}
\]

\[
\therefore \frac{f}{f_s} \cdot 2\pi = \frac{\# \text{ radians}}{\text{Sample}} = \left(\text{phase advance between}\right) \frac{\text{Samples}}{\text{Samples}}
\]

If \[\frac{f}{f_s} = \frac{1}{10},\] then there are 10 samples/snapshots per cycle.

\[\frac{f}{f_s} = \frac{1}{4}, \quad \frac{1}{2}, \quad 2\]

\[
t = \frac{T}{4}, \quad t = 0, T, \quad t = \frac{3T}{4}, \quad t = 0, T, \quad t = \frac{T}{2}, \quad t = 0, T, \quad t = \frac{T}{2}, \quad t = 0, T.
\]

We see the motion pretty well.

We can still see the motion. Seems not move.

If \[\frac{f}{f_s} \in \left(\frac{1}{2}, 1\right)\]

\[
t = \frac{T}{2}, \quad t = 0, T, \quad t > \frac{T}{2}, \quad t > T.
\]

Seems like going backward with a smaller frequency.

Seems like going forward with a much smaller frequency.
To summarize, if sampling rate is $f_s$

Sinusoidal signals with $f = 0, f_s, 2f_s, \ldots$ are ambiguous to each other.

If $0 < f < \frac{f_s}{2}$, it's ambiguous to $R = f - f_s$.

Mathematics:

$$X[n] = X(nT_s)$$

It's more convenient to represent this process in two stages:

$$S(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

Difference between $X_s(t)$ and $X[n]$:
1) $X_s(t)$ is a continuous-time signal while $X[n]$ is discrete-time signal.
2) $X_s(t)$ takes infinite values while $X[n] = X(nT)$ takes finite values.
3) $X[n]$ is the area under $X_s(nT)$, i.e. integration.

Now we can analyze the frequency content of $X_s(t)$ and compare it with $X(t)$. 
Therefore, if $X(t)$ contains frequency higher than $\frac{f_s}{2}$, i.e., $f_{\text{max}} > \frac{f_s}{2}$, then the replicas of $X(f)$ will overlap. This is called **aliasing**.

When aliasing happens, we will not be able to recover $X(f)$ from $X_s(f)$, i.e., not able to recover $X(t)$ from $X_s(t)$.

How to prevent aliasing?

1) Make sure $X(t)$ do not contain frequencies $> \frac{f_s}{2}$, called **Nyquist freq.**

   If it contains, apply an LP filter first.

   $X(f) \uparrow \quad \uparrow H(f)$

   Analog LP filter

   $\qquad \frac{-f_s}{2} \quad \frac{f_s}{2} \quad \frac{-f_s}{2} \quad \frac{f_s}{2} \quad f$

2) Sample fast enough, i.e., let $f_s \geq 2f_{\text{max}}$, called **Nyquist rate**.
Derivation of Fourier transform of impulse train

\[ S(t) = \sum_{n=\infty}^{\infty} \delta(t - nT_s) \]

So it has Fourier series expansion.

\[ S(t) = \sum_{m=\infty}^{\infty} C_m e^{j2\pi mt/T_s} \]

(a summation of periodic signals, i.e., complex sine waves)

\[ e^{+j2\pi t/T_s} \] is the fundamental sine

\[ e^{j2\pi mt/T_s} (m = \pm 2, \pm 3, \ldots) \] are harmonic

Remember that

\[ \int_{t_0}^{t_0+T_s} e^{j2\pi nt/T_s} \cdot e^{-j2\pi nt/T_s} dt = \begin{cases} 0, & \text{if } m \neq n \\ T_s, & \text{if } m = n \end{cases} \]

\[ \int_{t_0}^{t_0+T_s} e^{j2\pi nt/T_s} \cdot e^{-j2\pi nt/T_2} dt \\
\]

\[ C_m = \frac{1}{T_s} \int_{t_0}^{t_0+T_s} S(t) e^{-j2\pi mt/T_s} dt \]

\[ = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} S(t) e^{-j2\pi mt/T_s} dt \quad (\text{shift to another period}) \]

\[ = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-j2\pi mt/T_s} dt \quad (\text{only one impulse here}) \]

\[ C_m = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) dt = \frac{1}{T_s} \]

\[ S(t) = \sum_{n=\infty}^{\infty} \frac{1}{T_s} e^{j2\pi mt/T_s} \]

\[ S(f) = \int_{-\infty}^{\infty} S(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \sum_{n=\infty}^{\infty} \frac{1}{T_s} e^{j2\pi mt/T_s} e^{-j2\pi ft} dt \]

\[ = \frac{1}{T_s} \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi (f - mf_s)t} dt = \frac{1}{T_s} \sum_{m=\infty}^{\infty} \delta(f - mf_s) \]