System: \( x[n] \xrightarrow{T} y[n] \) Denote \( y[n] = T\{x[n]\} \)

**Linear Systems:** \( \text{iff} \quad T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} \quad \text{additivity} \)

\[ T\{a \cdot x[n]\} = a \cdot T\{x[n]\}, \quad \forall a \quad \text{Scaling} \]

\[ T\{a \cdot x_1[n] + b \cdot x_2[n]\} = a \cdot T\{x_1[n]\} + b \cdot T\{x_2[n]\}, \quad \forall a, b. \]

**Time-Invariant Systems:** (or **shift-invariant systems**)

For any input sequence \( x[n] \) and its time-shifted version \( x_1[n] = x[n-n_0], \forall n \)

if \( T\{x_1[n]\} = y[n-n_0], \) where \( T\{x[n]\} = y[n], \) then the system is time-invariant.

In other words, if the input is delayed, then the output is delayed for the same amount, but the shape remains.

(Note: Homework grading system is not time-invariant!)

**Linear Time-Invariant (LTI) Systems:**

Both linear and time-invariant.

Write \( x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \quad \text{(linearity)} \)

\[ y[n] = T\{x[n]\} = T\{\sum_{k=-\infty}^{\infty} x[k] \delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} \]

(time-invariant) \( \rightarrow \) \( \sum_{k=-\infty}^{\infty} x[k] h[n-k] \), where \( h[n] = T\{\delta[n]\} \)

\[ = x[n] * h[n] \]

\[ \text{An LTI System is completely characterized by its impulse response!} \]
Causality: A system is \textit{causal} if \( y[n_0] \) the output value at \( n_0 \), only depends on current and previous input values \( x[n] \) where \( n \leq n_0 \).

Stability: A system is \textit{stable} if for every bounded input, the output is bounded, i.e., \( |x[n]| \leq B_x < \infty \) for all \( n \), \( \exists B_y, B_y < \infty \) for all \( n \).

For LTI Systems: It is stable iff the impulse response is absolutely summable, i.e., \( B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty \) (not hard to prove).

For LTI Systems: Causal iff impulse response \( h[n] = 0 \), \( \forall n < 0 \)

Frequency Response: \( y[n] = x[n] * h[n] \)
\[
= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]
\]

If we let \( x[n] = e^{j\omega n} \), i.e., complex sinusoidal input sequence.
then \( y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j\omega(n-k)} \)
\[
= \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\omega k} \cdot e^{j\omega n}
= H(e^{j\omega}) \cdot e^{j\omega n} = H(e^{j\omega}) \cdot x[n]
\]

\( H(e^{j\omega}) \) is the frequency response of the system
\( H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\angle H(e^{j\omega})} \)

One way to get \( H(e^{j\omega}) \) for all \( \omega \): try all kinds of complex sinusoidal inputs with different frequencies, and calculate the difference between output and input.

Another way: Perform DTFT on impulse response \( h[n] \).
DTFT: \[ X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \]

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{(Replace } e^{jw} \text{ with } z) \]

This gives us the entire complex plane to analyze the system instead of just the unit circle.

The Z transform equation does not always converge. Depending on \( x[n] \), it has a Region of Convergence (ROC).

If ROC contains the unit circle, then DTFT converges (exists).

Properties of Z transforms (similar to those of Fourier transforms):
- Linearity: \[ a x_1[n] + b x_2[n] \overset{Z}{\leftrightarrow} a X_1(z) + b X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]
- Time shifting: \[ x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0} X(z), \quad \text{ROC } = R_x \]
- Convolution: \[ x_1[n] \ast x_2[n] \overset{Z}{\leftrightarrow} X_1(z) X_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2} \]

Z transform of LTI Systems.
\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \]
\[ Y(z) = H(z) \cdot X(z) \quad \text{System function} \]

Since \( h[n] \overset{Z}{\leftrightarrow} H(z) \), ROC form a unique pair.

\( H(z) \) completely characterizes the LTI system.

If ROC contains the unit circle, then \( H(e^{jw}) \) exists, and it also completely characterizes the system.
A general LTI filter represented by difference equation.

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l] \]

Take Z transform: \( \sum_{k=0}^{N} a_k Z^{-k} \cdot Y(z) = \sum_{l=0}^{M} b_l Z^{-l} \cdot X(z) \)

\[ \therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l Z^{-l}}{\sum_{k=0}^{N} a_k Z^{-k}} \left\text{polynomial of order } M \right. \]

\( \left( \text{degree} \right) \)

(rewritten as)

\[ = \frac{b_0}{a_0} \prod_{k=1}^{M} \left(1 - c_k Z^{-1}\right) \left\text{M complex roots.} \right. \]

\[ = \prod_{k=1}^{N} \left(1 - d_k Z^{-1}\right) \left\text{M complex roots, counted with multiplicity.} \right. \]

[Fundamental Theorem of Algebra: Any single variable polynomial of order \( M \) has \( M \) complex roots, counted with multiplicity.]

Also, any non-real root must have its complex conjugate also being a root.

\[ \therefore \text{every term } (1 - c_k Z^{-1}) \text{ in the numerator contributes a zero at } Z = c_k \text{ and a pole at } Z = 0. \]

\[ \text{every term } (1 - d_k Z^{-1}) \text{ in the denominator contributes a pole at } Z = d_k \text{ and a zero at } Z = 0. \]

IIR filter: \( h[n] \) has infinitely many nonzeros.
If \( H(z) \) contains at least one nonzero pole, then IIR.

FIR filter: \( h[n] \) has finite nonzeros.
If all the poles of \( H(z) \) are zeros.
If \( H(z) \) has no pole other than \( Z=0 \), then FIR.
A simple FIR filter

\[ Y[n] = X[n] + a_1 X[n-1] \]

\[ Y(z) = X(z) + a_1 z^{-1} X(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} = 1 + a_1 z^{-1} = \frac{z + a_1}{z} \]

1-zero : \( z = -a_1 \), 1 pole : \( z = 0 \).

FIR filter

\[ |H(z)| = \left| \frac{z + a_1}{z} \right| \]

Frequency response:

\[ |H(e^{j\omega})| = \left| \frac{e^{j\omega} + a_1}{e^{j\omega}} \right| = |e^{j\omega} - (-a_1)| \]

(distance between \( e^{j\omega} \) and zero.)

\[ |H(e^{j\omega})| \]

if \( a_1 > 0 \)

Low-pass filter.

If \( a_1 = 1 \), then \( |H(e^{j\omega})| = 0 \) when \( \omega = \pi \)

If \( a_1 = 0 \), then \( |H(e^{j\omega})| = 1 \), All pass filter

If \( a_1 < 0 \), then high-pass filter
A second-order FIR filter

\[ y[n] = x[n] + x[n-1] + x[n-2] \]

\[ Y(z) = X(z) - z^{-1} X(z) + z^{-2} X(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1} + z^{-2} = \frac{z^2 - z + 1}{z^2} \]

Poles: \( z = 0 \) twice.

Zeros: roots of \( z^2 - z + 1 = 0 \) \( \Rightarrow \) \( z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm j \frac{\sqrt{3}}{2} = e^{\pm j\frac{\pi}{3}} \)

\[ H(z) = \frac{z^2 - z + 1}{z^2} = \frac{(z - e^{j\frac{\pi}{3}})(z - e^{-j\frac{\pi}{3}})}{z^2} \]

\[ |H(e^{j\omega})| = |e^{j\omega} - e^{j\frac{\pi}{3}}||e^{j\omega} - e^{-j\frac{\pi}{3}}| \]

When \( \omega = 0 \), \( |H(e^{j\omega})| = 1 \cdot 1 = 1 \)

When \( \omega = \frac{\pi}{3} \), \( |H(e^{j\omega})| = 0 \)

When \( \omega = \pi \), \( |H(e^{j\omega})| = \sqrt{3} \cdot \sqrt{3} = 3 \)

\[ |H(e^{j\omega})| \]
In general, LTI causal FIR filter:

\[ y[n] = \sum_{i=0}^{M} b_i x[n-i] \]

order: \( M \) (taps).
zeros: \( M \) zeros
poles: \( M \) poles all at \( z=0 \).

impulse response:

\[ h[k] = b_k, \quad 0 \leq k \leq M \]
\[ 0, \quad \text{otherwise} \]

A simple filter with a feedback loop:

\[ y[n] = x[n] + a_1 y[n-1] \]

impulse response: Let \( x[n] = \delta[n] \), initialize \( y[-1] = 0 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( a_1^2 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( a_1^3 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

How long does it take for \( y[n] \) to decay less than quantization noise level?

\[ a_1^k < 2^{-N/2} \]

\[ k \log_2 a_1 < \frac{N}{2} \]

\[ k > \frac{N}{\log_2 a_1} \]

Suppose \( a_1^k < 2^{-N/2} \) if \( N = 16 \) bit
\( a_1 = 0.9 \)
then \( k > 98.7 \) 105.3
Apply $z$ transform: $Y(z) = X(z) + a_{1}z^{-1}Y(z)$

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a_{1}z^{-1}} = \frac{z}{z - a_{1}} \]

One zero at $z = 0$, one pole at $z = a_{1}$.

Frequency response $H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a_{1}}$

\[ |H(e^{j\omega})| = \frac{1}{|e^{j\omega} - a_{1}|} \]

When $\omega = 0$, $|H(e^{j\omega})| = \frac{1}{|1 - a_{1}|}$

$\omega = \pi \Rightarrow \frac{1}{|1 + a_{1}|}$

if $a_{1} = 0.5$.

Clearly, if $0 < a_{1} < 1$, LPF

- $a_{1} = 0$, All pass filter, $y[n] = x[n]$
- $-1 < a_{1} < 0$, HPF
- $|a_{1}| \geq 1$, non-stable $\sum_{n=0}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a_{1}|^{n}$ diverges.

A second order IIR filter:

\[ y[n] = x[n] + 0.64 \cdot y[n-2] \]

\[ Y(z) = X(z) + 0.64z^{-2}Y(z) \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.64z^{-2}} = \frac{z^{2}}{z^{2} - 0.64} \]

\[ = \frac{z^{2}}{(z - 0.8)(z + 0.8)} \]

Zeros: $z = 0$ twice
Poles: $z = 0.8$, $z = -0.8$

\[ |H(e^{j\omega})| = \frac{1}{|e^{j\omega} + 0.8||e^{j\omega} - 0.8|} \]
A general LTI, causal, all-pole, IIR filter

\[ y[n] = b_0 x[n] + \sum_{l=1}^{N} a_l y[n-l] \]

\[ Y(z) = b_0 X(z) + \sum_{l=1}^{N} a_l z^{-l} Y(z) \]

\[ H(z) = \frac{b_0}{1 - \sum_{l=1}^{N} a_l z^{-l}} \]

- \(N\) zeros, all at \(z=0\)
- \(N\) poles must be all within unit circle
  - otherwise unstable.

---

A general LTI, causal, IIR filter

\[ y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{l=1}^{M} a_l y[n-l] \]

\[ Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k} - \sum_{l=1}^{M} a_l z^{-l} Y(z) \]

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{l=1}^{M} a_l z^{-l}} \]

- \(M\) non-zero zeros
- \(N\) poles
Phase response: \[ H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})} \]

\[ \angle H(e^{j\omega}) = \tan^{-1} \frac{\text{Im} H(e^{j\omega})}{\text{Re} H(e^{j\omega})} \]

Look at the simplest FIR filter again:

\[ y[n] = x[n] + b_1 x[n-1] \]

\[ y(z) = 1 + b_1 z^{-1} \]

\[ H(z) = \frac{1 + b_1 z^{-1}}{1 + b_1 z^{-1}} = 1 + b_1 e^{-j\omega} = 1 + b_1 \cos \omega - j b_1 \sin \omega \]

\[ \therefore \angle H(e^{j\omega}) = \tan^{-1} \frac{-b_1 \sin \omega}{1 + b_1 \cos \omega} \]

If \( b_1 = 1 \), then \( \tan^{-1} \frac{-\sin \omega}{1 + \cos \omega} = \tan^{-1} \frac{-2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}} = \tan^{-1} \left( -\frac{\omega}{2} \right) = -\frac{\omega}{2} \)

i.e., half-sample delay.

Look at impulse response:

\[ x[n] \]

\[ n \]

\[ y[n] \]

\[ n \]

\[ H(z) = g \frac{z^{N-M} \sum_{k=1}^{M} (z-C_k)}{\sum_{k=1}^{M} (z-D_k)} \]

\[ H(e^{j\omega}) = g e^{j(N-M)\omega} \prod_{l=1}^{M} A_l e^{j\phi_l} \prod_{k=1}^{N} B_k e^{j\theta_k} \]

\[ \therefore \angle H(e^{j\omega}) = \sum_{l=1}^{M} \phi_l - \sum_{k=1}^{N} \theta_k + (N-M)\omega \]