1. Perspective on Two-Port Electromechanical Systems

Lossless system: without loss of generality, all losses are incorporated into electrical and mechanical port.

\[ P_e = P_m \]

or

\[ P_e = P_m \frac{d}{dt} \Rightarrow dW_e = dW_m \]

In general, we have initial energy accumulation in addition to the energy transfer. Thus, for the actuator/motor system of Figure 1(a) one can write

\[ \frac{dW_e}{dt} = \frac{dW_e}{dt} + \frac{dW_a}{dt} \]

For mechanical systems the power and energy increment are given by

\[ P_m = f \cdot (dx/dt) \]

\[ dW_m = f \cdot (dx/dt)dt \]
For electrical systems the power and energy increments are

\[ P_e = v \cdot i \]
\[ dW_e = v \cdot idt \]

There are two special cases of interest for electrical energy increments:

1. electrostatic case
   \[ dW_e = v \cdot idt = v \cdot (dq/dt)dt = vdq \]

2. magnetostatic case
   \[ dW_e = v \cdot idt = (d\lambda/dt) \cdot idt = id\lambda \]

Although the following derivation applies to both cases we will frequently refer to the electrostatic case to illustrate the theory. Magnetostatic examples will follow later, to reinforce our understanding.

To formulate a mathematical model for the system of Figure 1, we need to relate four quantities: two mechanical (\( f \) and \( x \)) and two electrical (\( v \) and \( i \)). Thus, in order to solve the mathematical model, we need total of four equations. Note, however, that there is a constraint equation for the mechanical port and a constraint equation for electrical port. Circuit theory provides constraint equation for electrical constraints and mechanical engineering provides equations for mechanical constraints. At this point we understand that these equations exist and take them implicitly. Our objective here is to formulate the remaining two equations. This formulation is quite general and does not depend upon the electrical and mechanical constraints. Once derived, these two equations can be coupled to the two constraint equations and the four equations can be solved for a particular case at hand.
2. Equations for a General Electromechanical Two-Port

Before we derive our equations it is important to make a few general observations. A lossless electromechanical system is an accumulative (time-evolving) system. An accumulative system cannot be characterized by algebraic equations. Typically, one needs to formulate a set of differential equations including differential equations of the constraints and solve them simultaneously. This approach is widely used, but it is not very general, as it has to be formulated for each set of constraints...

Consider again accumulated energy in a two-port actuator/motor system. From the black-box description of Figure 1 we have

\[ dW_a = dW_e - dW_m \]

To proceed further we look at electrostatic systems

\[ dW_a = v dq - f dx \]

The four terminal variables \((v, i, f, \text{ and } dx/\text{dt})\) are not mutually independent. The two constraining equations relate electrical variables to one another and mechanical variables to one another. We have two independent variables: one mechanical and one electrical. The equation above suggests selecting \(q\) and \(x\) as our independent variables. To show that explicitly we rewrite the equation as follows

\[ dW_a(q,x) = v(q,x) dq - f(q,x) dx \]

To find \(W_a\) we need to integrate the previous equation. Here we take advantage of the properties of the lossless (conservative) systems. These systems do not depend upon the path of integration (see figure below).

**Figure:** A conservative system is path independent. Note the two special (easy) paths.

Path 1:

\[
W_a = \int_0^x \frac{\partial W_a}{\partial x}(x,0) \, dx + \int_0^q \frac{\partial W_a}{\partial x}(x_0,q) \, dq = -f(x,0) = v(x_0,q)
\]

Path 2:

\[
W_a = \int_0^q \frac{\partial W_a}{\partial q}(0,q) \, dq + \int_0^x \frac{\partial W_a}{\partial x}(x,q_0) \, dx = v(0,q) = -f(x,q_0)
\]
To take advantage of the path-independence, we select the most convenient (easiest) path for integration. It is useful to rewrite $dW_a$ as a total differential of $W_a$

$$dW_a = \frac{\partial W_a}{\partial q} dq + \frac{\partial W_a}{\partial x} dx$$

Comparing the two $dW_a$ equations, one can see that

$$v(x, q) = \frac{\partial W_a(x, q)}{\partial q} \quad \text{and} \quad f(x, q) = -\frac{\partial W_a(x, q)}{\partial x}$$

Path 1 and Path 2 are the best candidates, because they separate integration over $q$ from integration over $x$. Path 1 is actually the best, because force of electric origin is zero before the charge is accumulated. Thus

$$W_a(x_o, q_o) = \int_0^{q_o} v(x_o, q) \, dq$$

For a linear capacitance we have

$$v(x_o, q_o) = q_o/C(x_o) \quad \text{– the dependence on } x \text{ is contained in the capacitance } C(x)$$

Inserting $v(x_o, q_o)$ into expression for $W_a(x_o, q_o)$ and carrying the integration gives us

$$W_a(x_o, q_o) = -\frac{q_o^2}{C(x_o)}$$

Using the relations comparing the two $dW_a$ equations the expression for the force and voltage directly follow

$$f = -\frac{\partial W_a(x, q)}{\partial x} \bigg|_{x = x_o} = \frac{1}{2} \frac{q_o^2}{C^2(x = x_o)} \frac{dC(x = x_o)}{dx}$$

$$v = \frac{\partial W_a(x, q)}{\partial q} \bigg|_{x = x_o} = \frac{q_o}{C(x_o)}$$

Important Note:

Here, we take partial derivatives of the function $W_a(x, q)$. Thus if we rearrange $W_a$ and express it as $W_a = C(x)v^2$, we have to be very careful about taking the derivatives because this is a different function $W_a(x, v(x, q))$. If we want to find the force using the expression above, we must rely on chain rule differentiation:
The functions \(v(x,q)\) and \(f(x,q)\) are nonlinear functions of their arguments. It is often very useful to linearize these functions.

Digression: A beautiful example of the usefulness of linearization concerns the flat-earth and round-earth models. The round-earth model is more accurate, but in everyday life we typically use the flat plane model with adequate success to get around. [Fonstad C. G. Microelectronic Devices and Circuits, McGraw-Hill Inc., 1994 pp. 2-3]

Linear operation of electromechanical system is frequently a requirement in sensor design to minimize distortion in the sensed signal.

Linearizing the force and voltage equation

\[
\begin{align*}
\frac{df}{dx} + \frac{df}{dq} dq & \to \ddot{f} = - \frac{\partial^2 W_a}{\partial x^2} \dot{x} + \frac{\partial^2 W_a}{\partial q \partial x} \dot{q} \\
\frac{dv}{dx} + \frac{dv}{dq} dq & \to \ddot{v} = \frac{\partial^2 W_a}{\partial x \partial q} \dot{x} + \frac{\partial^2 W_a}{\partial q^2} \dot{q}
\end{align*}
\]

For linear system we can use complex domain where a time-variable \(a(t)\) is replaced by a complex-variable \(a^*\) with \(a(t) = \text{Re}\{a^* e^{j\omega t}\}\). From this we have \(d/dt\) \(\leftrightarrow j\omega(\cdot)\)

\[
\begin{align*}
\dot{f} & = \left( - \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial x^2} \right) (j\omega \dot{x}) + \left( - \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial q \partial x} \right) j\omega \dot{q} \\
\dot{v} & = \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial x \partial q} j\omega \dot{x} + \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial q^2} j\omega \dot{q}
\end{align*}
\]

These equations are formulated for an actuator. For a general transducer it is convenient to reverse the sign of mechanical power so that both ports insert power into the system. This can be done by reversing the sign of the displacement: \(x \to -x\)
\[ \hat{f} = \left( \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial x^2} \right) (j\omega \hat{x}) + \left( -\frac{1}{j\omega} \frac{\partial^2 W_a}{\partial q \partial x} \right) \hat{i} \]

\[ = Z_m \]

\[ \hat{v} = \left( -\frac{1}{j\omega} \frac{\partial^2 W_a}{\partial x \partial q} \right) j\omega \hat{x} + \left( \frac{1}{j\omega} \frac{\partial^2 W_a}{\partial q^2} \right) \hat{i} \]

\[ = -M^* = M \]

The last two equations can be compactly rewritten in matrix form

\[ \begin{bmatrix} \hat{f} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} Z_m & M \\ -M^* & Z_e \end{bmatrix} \begin{bmatrix} j\omega \hat{x} \\ \hat{i} \end{bmatrix} \]

As mentioned earlier, the constraint equation can be inverted, and it is sometimes convenient to have a different set of independent variables. To this end we use here the Legendre’s transformation and concept of coenergy.

**Digression:** Legendre’s transformation is an important concept. For example, it is also used in analytical dynamics for converting Lagrangian into Hamiltonian and vice versa [see, for example, Lanczos C *The Variational Principles of Mechanics*, Dover Publication, the Fourth edition 1986 pp 163-164]

Instead of \((x, q)\) as independent variables, it is often more convenient to choose \((x, v)\). To achieve this, we define coenergy as

\[ W_a^c = vq - W_a \quad \Rightarrow \quad dW_a^c = qdv + fdx \]

Then we choose the most convenient path to find \(W_a^c\)

\[ W_a^c(x_o, v_o) = \int_{x_o}^{x_o} f(x, 0) dx + \int_{v_o}^{v_o} q(x_o, v) dv \]

For a linear electrostatic case \(q(x, v) = C(x)v\) and

\[ W_a^c(x_o, v_o) = _{-}C(x_o)v_o^2 \]

It is important to understand that energy and coenergy are in general different functions. For a case of a linear capacitance they just happen to be the same. See the figure bellow.
Digression: Also, one can define another energy functions to express energy in terms of the variables:

\[ W_{ac}^{c2} = -f_x - W_a \Rightarrow dW_{ac}^{c2}(f, q) = -\partial W_{ac}^{c2}(f, q)/\partial f df - \partial W_{ac}^{c2}(f, q)/\partial q dq \]

Or

\[ W_{ac}^{c3} = -f_x + vq - W_a \Rightarrow dW_{ac}^{c3}(f, v) = -\partial W_{ac}^{c3}(f, v)/\partial f df + \partial W_{ac}^{c3}(f, v)/\partial q dv \]

Using the same linearization procedure we obtain

\[
\hat{f} = \left(-\frac{1}{j\omega} \frac{\partial^2 W_{ac}^{c}}{\partial x^2}\right) (j\omega \hat{x}) + \frac{\partial^2 W_{ac}^{c}}{\partial q \partial x} \hat{y} = Z_m^c \nabla_c^2 (\hat{x}) = N
\]

\[
\hat{i} = -\frac{\partial^2 W_{ac}^{c}}{\partial x \partial q} j\omega \hat{x} + \frac{1}{j\omega} \frac{\partial^2 W_{ac}^{c}}{\partial v^2} \hat{v} = -N^* = \dot{N} = Y_e
\]

or in matrix form

\[
\begin{bmatrix}
\hat{f} \\
\hat{i}
\end{bmatrix} = \begin{bmatrix}
Z_m^c & N^* \\
-N^* & Y_e
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix}
\]

Note that \( Z_e, Z_m, Y_e, \) and \( Z_m^c \) are all purely imaginary numbers. This is expected because the electromechanical two-port is assumed to be lossless.

Linearized equations can be equivalently represented by equivalent circuits.
\[
\begin{bmatrix}
\hat{f} \\
\hat{v}
\end{bmatrix} =
\begin{bmatrix}
Z_m & M \\
-M^* & Z_e
\end{bmatrix}
\begin{bmatrix}
\hat{j} \\
\hat{i}
\end{bmatrix} \leftrightarrow
\begin{bmatrix}
\hat{f} \\
\hat{i}
\end{bmatrix} =
\begin{bmatrix}
Z^c_m & N \\
-N^* & Y_e
\end{bmatrix}
\begin{bmatrix}
\hat{j} \\
\hat{v}
\end{bmatrix}
\]

Relationship between \( M \) and \( N \) form: convert Thevenin source on the electrical side into Norton source:

\[
Y_e = \frac{1}{Z_e} \quad -N'(j\omega x) = -M'(j\omega x)/Z_e \quad \rightarrow \quad N = \frac{M}{Z_e}
\]

Now use this in the mechanical port:

\[
M_i = M\left(-N^* j\omega x + Y_e v\right) = M\left(\frac{-M^*}{Z_e} j\omega x + \frac{1}{Z_e} v\right) = \frac{|M|^2}{Z_e} j\omega x + \frac{M}{Z_e} v
\]

\[
\frac{Z^c_m - Z_m}{Z_e} = N
\]

Important Examples:

1. Variable-gap capacitance
   - Energy; constant charge; \( M \) – form
   - Constant charge is autonomous system (minimizes energy)...
   - MEMS sensor because a small motion can make a difference

2. Variable-area capacitance
   - Coenergy; constant voltage; \( N \) – form
   - Constant voltage is non-autonomous system (energy comes from the battery)
   - MEMS actuators because it can have constant force over large motion

Multi-port Examples:

Two electrical and one mechanical; two- mechanical and one electrical

Application of constraints; bode diagrams etc.

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